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SUBCRITICAL STATES OF ASYMMETRIC NUCLEAR MATTER[a]

The isobaric caloric curve is considered in the subcritical states region. The energy fluctuations along the caloric curve are determined for small nuclear systems that consist of a limited number of nucleons. The temperature dependence of heat capacity at fixed pressure is obtained. The calculated quantities of the small nuclear system are discussed and checked against those for the infinite nuclear matter.

Keywords: caloric curve, energy fluctuation, small nuclear system.

1. Introduction

Caloric curves measurements in heavy-ion collisions [1–3] have shown signs of a liquid-vapor phase transition in nuclear matter. The caloric curve, that is, the dependence of temperature, T, on excitation energy per particle ε_{ex}, provides important information on the equation of state of nuclear matter. The plateau region of the caloric curve, where one observes a small change in temperature within the wide range of excitation energy, gives the signal of a phase transition [4]. This flat region of the caloric curve is accompanied by an increase in energy fluctuations and demonstrates a departure from the Fermi-gas-like equation of state. For finite nuclear systems, the extension of concepts as applied in the case of infinite matter is possible [5]. Statistical mechanics allows us to find signatures of liquid-vapor phase transition for finite nuclear systems composed of a limited number of neutrons N and protons Z (small systems) [6]. Generally, the liquid-vapor phase transition is accompanied by an increase of energy fluctuations. In this context, it is of interest to consider fluctuations for thermodynamic states along the caloric curve. Here, we address this issue within the Isothermal-Isobaric Ensemble formulation. Calculations are performed at subcritical (i.e., below critical) values of temperature T and pressure P.

Theoretical study of the infinite nuclear matter [4] shows that the equation of state for nuclear matter exhibits behavior like the van der Waals equation of state, having a spinodal region with negative incompressibility. Nuclear matter can evolve through phase separation boundaries and exhibit a liquid-vapor phase transition. Finite nuclear systems at excitation energies starting from several tens of megaelectron-volt per nucleon (about of the Fermi energy) show a similar behavior accompanied by large energy fluctuations. The link of the liquid-vapor phase transition with the multifragmentation for finite nuclear systems can be demonstrated based on the study of the caloric curve [7]. The intermediate nuclear system generated in heavy ion collision is considered as a piece of hot dense nuclear matter which expands, cools down and breaks up into separate clusters and light particles. Final cold products are observed experimentally. Despite of somewhat different scenarios of multifragmentation, see, for example, [8], at least partial thermodynamical equilibrium of the intermediate system is assumed at some stage. This allows us to introduce the thermodynamical quantities like temperature, T, and consider the system within the statistical mechanics using the partition sum for the statistical weights calculation of various break-up channels rather than in terms of kinetic equations for the detailed dynamics of the multifragmentation process.

Statistical mechanics is not restricted to infinite systems and can be applied to a collection of a small number of particles. Below, we assume that the fragments of intermediate nuclear systems are the pieces of nuclear matter at some temperature surrounded by nucleon vapor and other fragments (clusters) at equilibrium, and thermodynamic functions for these fragments are calculated from statistical mechanics. Here, we go over the consideration of the ensemble of clusters [6] calculating the average properties, like, for example, energy, as the corresponding ensemble average. Similarly, when performing the experiment,

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one does not make measurements on a single cluster, obtaining the averages over the flux of detected clusters. Few tens or hundreds of particles cannot, obviously, be considered as infinite matter unless as in the first crude approximation. One of the goals of this paper is to highlight the difference between the infinite nuclear matter and the small nuclear system, making emphasis on the liquid-vapor phase transition well known for the infinite matter.

In Section 2, the definitions of partition sum and Gibbs free energy are considered. The energy and enthalpy fluctuations are obtained. Section 3 is devoted to phase equilibrium in asymmetric nuclear matter. The relations for fractions of the total number of nucleons redistributed between liquid and vapor phases are determined. The shapes of the caloric curves and specific heat are obtained in Section 4 for small nuclear systems as well as for infinite nuclear matter. Energy fluctuations for small systems are also calculated. Concluding remarks are summarized in Section 5.

2. Partition sum and Gibbs free energy. Energy fluctuation

In the macroscopic limit of very large volume and particle number (within the habitual thermodynamics for infinite nuclear matter), all the ensembles give the same description of energy as a linear homogeneous function of entropy, volume, and particle number. This is not the case for the small systems of a limited number of particles, even though the surface, curvature, and Coulomb effects are neglected. Nevertheless, for a certain set of environment variables, like pressure, *P*, and temperature, *T*, in our case, the thermodynamics of a small system can be built [6]. In this paper, we consider caloric curves at a fixed value of pressure. In order to describe isobaric caloric curves, it is convenient to use Isothermal-Isobaric Ensemble formalism.

The partition sum Δ of the Isothermal-Isobaric Ensemble is given by

$$\Delta(N,Z,P,T)=\sum_{i,V}\exp\left[\left(-PV-E_i\right)/T\right]=$$

$$=\sum_{V}\exp\left[\left(-PV-F(N,Z,V,T)/T\right]\right], \qquad (1)$$

where *P* is the pressure, $\{E_i\}$ is the energy spectrum of a small nuclear system. In the right-hand side of Eq. (1), the summation (integration) is carried out over volume *V*, and $F = F(N, Z, V, T)$ stands for the free energy of a nuclear system consisting of *N* neutrons and *Z* protons. It is assumed that the free energy *F* can be scaled to given particle composition $A = N + Z$ from the Thomas - Fermi free energy per particle $\phi_{\mathrm{TF}}\left(\rho_n,\rho_p,T\right)$ [4] as $F=A\phi_{\mathrm{TF}}\left(\rho_n,\rho_p,T\right)$, where $\rho_n = N/V$ and $\rho_p = Z/V$ are, respectively, the neutron and proton densities. The thermodynamic potential of the Isothermal-Isobaric Ensemble, that is Gibbs free energy *G*, is written as

$$G(N,Z,P,T)=-T\ln\Delta(N,Z,P,T). \qquad (2)$$

As seen from (2), the pressure and temperature are natural environment variables of Gibbs free energy. In order to account for the situation that the system can be found in a two-phase thermodynamic state (liquid + vapor), the partition sum yields

$$\Delta(N,Z,P,T)=$$

$$=\sum_{\substack{N^{\mathrm{liq}}+N^{\mathrm{vap}}=N,\\ Z^{\mathrm{liq}}+Z^{\mathrm{vap}}=Z,\\ V^{\mathrm{liq}},V^{\mathrm{vap}}}}\exp\left[\left(-P\left(V^{\mathrm{liq}}+V^{\mathrm{vap}}\right)-F^{\mathrm{liq}}-F^{\mathrm{vap}}\right)/T\right], \qquad (3)$$

where $F^{\mathrm{liq}}=F\left(N^{\mathrm{liq}},Z^{\mathrm{liq}},V^{\mathrm{liq}},T\right)$, $F^{\mathrm{vap}}=F\left(N^{\mathrm{vap}},Z^{\mathrm{vap}},V^{\mathrm{vap}},T\right)$, and superscripts “liq” and “vap” denote liquid and vapor phases, respectively. The partition sum in Eq. (3) takes the account of particles redistributed between liquid and vapor phases, provided the total number of neutrons and protons is fixed. The average value of energy, $\langle E\rangle$, and its dispersion (the energy fluctuation σ_E squared), $\sigma_E^2=\langle E^2\rangle-\langle E\rangle^2$, are obtained from the Gibbs free energy *G* as

$$\langle E\rangle = G-T\left(\frac{\partial G}{\partial T}\right)_{N,Z,P}-P\left(\frac{\partial G}{\partial P}\right)_{N,Z,T}, \qquad (4)$$

and

$$\langle E^2\rangle-\langle E\rangle^2=-T^3\left(\frac{\partial^2 G}{\partial T^2}\right)_{N,Z,P}-$$

$$-2PT^2\left(\frac{\partial^2 G}{\partial P\partial T}\right)_{N,Z}-P^2T\left(\frac{\partial^2 G}{\partial P^2}\right)_{N,Z,T}. \qquad (5)$$

One should note that the relation (4) connecting the thermodynamic functions and obtained here from (2) is applicable both for the infinite nuclear matter and the small nuclear system. However, the thermodynamic functions themselves differ. As a result, in the case of infinite matter, $A\to\infty$, the intensive quantity $\langle E\rangle/A$ does not depend on the particle number *A*, it does not matter what ensemble is used for the estimate, while the corresponding quantity for the case of the small system remains *A*-dependent, and the choice of the ensemble and its environmental variables does matter. The link between ordinary macroscopic thermodynamics and the thermodynamics of small systems can be achieved by considering a sample of independent small systems as a starting point [6]. The excitation energy per particle $\varepsilon_{\rm ex}$, needed for the determination of the caloric curve, $T(\varepsilon_{\rm ex})$, is obtained from Eq. (4) as

$$\varepsilon_{\rm ex} = \left(\langle E\rangle - E_{gs}\right)/A\,, \qquad (6)$$

where E_{gs} is the ground state energy at $T = 0$ and a fixed value of P.

Combined with the energy fluctuation, it is worthwhile to consider also the fluctuation of the enthalpy $\langle H\rangle = \langle E\rangle + P\langle V\rangle = G - T\left(\partial G/\partial T\right)_{N,Z,P}$ since the enthalpy is a natural heat function for variables P and T. The dispersion of the enthalpy $\sigma_H^2 = \langle H^2\rangle - \langle H\rangle^2$, is directly concerned with the heat capacity at fixed pressure, C_p, as follows [9]

$$\langle H^2\rangle - \langle H\rangle^2 = T^2\left(\frac{\partial\langle H\rangle}{\partial T}\right)_{N,Z,P} = C_P T^2. \qquad (7)$$

The fluctuation σ_H and heat capacity C_p become very large in a two-phase region at constant pressure and temperature due to the fact that heat can be absorbed, converting some of one phase into the other. So, a strong increase in heat capacity is an additional sign that a phase transition occurs.

3. Infinite matter

To highlight the difference of a small nuclear system against infinite matter, one needs to consider the macroscopic limit of a large nucleon number. Let's take a look at a nuclear system having so large particle number $A = N + Z$ that makes possible the use of regular thermodynamics. The Gibbs free energy per particle $g(P,T,X) = G/A = \left(\partial G/\partial A\right)_{P,T,X}$, where $X = (N-Z)/A$ is the asymmetry parameter, is a function of intensive variables only. Introducing isovector $\mu_0 = g - X\left(\partial g/\partial X\right)_{P,T}$, and isoscalar $\mu_1 = \left(\partial g/\partial X\right)_{P,T}$, chemical potentials, the Gibbs conditions for liquid-vapor equilibrium [10] are written as

$$\mu_0^{\rm liq}\left(P,T,X^{\rm liq}\right) - \mu_0^{\rm vap}\left(P,T,X^{\rm vap}\right) = 0\,, \qquad (8)$$

$$\mu_1^{\rm liq}\left(P,T,X^{\rm liq}\right) - \mu_1^{\rm vap}\left(P,T,X^{\rm vap}\right) = 0. \qquad (9)$$

We note that within the macroscopic limit, there is equivalence of Isothermal-Isobaric and Canonical Ensembles. The pressure in Eqs. (8) and (9), which should be the same for liquid and vapor phases in equilibrium, is associated with the equation of state $P = P(\rho, X, T)$ for infinite nuclear matter [11], where $\rho = A/V$ is the nucleon density. With regard to the mentioned equation of state, one has two solutions of Eqs. (8) and (9), which are $\left\{\rho^{\rm liq}, X^{\rm liq}\right\}$ and $\left\{\rho^{\rm vap}, X^{\rm vap}\right\}$, at a certain value of temperature T.

For the two-phase part of the isobar $(P = const)$ the energy per particle, $\varepsilon = E/A$, is determined as the sum of contributions from each phase,

$$\varepsilon = \alpha^{\rm liq}\varepsilon^{\rm liq} + \alpha^{\rm vap}\varepsilon^{\rm vap}\,, \qquad (10)$$

where $\alpha^{\rm liq} = A^{\rm liq}/A$ and $\alpha^{\rm vap} = A^{\rm vap}/A$ are the fractions of the total number of particles A, redistributed between liquid and vapor phases, respectively. From the conservation conditions for the total number of particles and neutron excess of the whole system determined by the particle A and asymmetry parameter X, namely,

$$\alpha^{\rm liq} + \alpha^{\rm vap} = 1\,, \quad \alpha^{\rm liq}X^{\rm liq} + \alpha^{\rm vap}X^{\rm vap} = X\,, \qquad (11)$$

one obtains the values of particle fractions:

$$\alpha^{\rm liq} = \left(X^{\rm vap} - X\right)/\left(X^{\rm vap} - X^{\rm liq}\right),$$

$$\alpha^{\rm vap} = \left(X - X^{\rm liq}\right)/\left(X^{\rm vap} - X^{\rm liq}\right)\,. \qquad (12)$$

By virtue of equilibrium conditions (8) and (9) the above particle fractions are, in general, P- and T-dependent. In the case of a single phase, one has to put $\alpha^{\rm liq} = 1$, $\alpha^{\rm vap} = 0$, $X^{\rm liq} = X$ for the liquid or $\alpha^{\rm liq} = 0$, $\alpha^{\rm vap} = 1$, $X^{\rm vap} = X$ for the vapor phase. The construction of the caloric curve requires the calculation of the excitation energy per particle $\varepsilon_{\rm ex}$. The value of $\varepsilon_{\rm ex}(T,P,X)$ is obtained by subtracting the corresponding ground state value of energy per particle $\varepsilon(T=0,P,X)$ from the corresponding value (10) at a certain value of $T > 0$:

$$\varepsilon_{\rm ex}(T,P,X) = \varepsilon(T,P,X) - \varepsilon(T=0,P,X). \qquad (13)$$

Similarly to the energy per particle of the asymmetric nuclear matter, we determine the value of enthalpy per particle $h = H/A = \varepsilon + P/\rho$, where $H = E + PV$ is the enthalpy. Just as in Eq. (10), the enthalpy per particle for the two-phase equilibrium state is given by

$$h = \alpha^{\text{liq}} h^{\text{liq}} + \alpha^{\text{vap}} h^{\text{vap}}. \quad (14)$$

Relying on Eq. (14), one can obtain the specific heat per particle at fixed pressure, $c_P = C_P / A$, which is defined by

$$c_P = \left(\frac{\partial h}{\partial T} \right)_{P,X}. \quad (15)$$

In contrast to the case of a single phase, the enthalpy per particle in the form of Eq. (14) contains particle fractions $\alpha^{\text{liq, vap}}$ which themselves are temperature dependent. This usually leads to an increase in the value of the specific heat as compared to the corresponding single-phase value. In order to obtain the temperature dependence of particle fractions, $\alpha^{\text{liq, vap}}(T)$ one may differentiate equilibrium conditions (8) and (9) with respect to temperature, assuming that their solutions for $X^{\text{liq, vap}}$ depend on T. As a result one arrives at two linear equations with regard to the derivatives $\left(\partial X^{\text{liq, vap}} / \partial T\right)_{P,X}$. Then, resolving the obtained equations, the temperature derivatives of particle fractions $\left(\partial \alpha^{\text{liq, vap}} / \partial T\right)_{P,X}$ can be found straightforwardly using relations (12).

4. Results and discussion

We study the shape of the caloric curve and energy fluctuations along this curve for the case of isobaric heating. For this purpose, we follow the temperature-dependent Thomas - Fermi theory using the Skyrme-type force as the effective nucleon-nucleon interaction [4, 12], see also Appendix A. We have calculated the isobaric caloric curves, using $P = 0.05\,\text{MeV/fm}^3$, for several small nuclear systems having different particle numbers $A = 48, 120,$ and 216 at the same asymmetry parameter $X = 1/6$. By the neutron-proton composition, these systems correspond to ^{48}Ca, ^{120}Sn, and ^{216}Th nuclei. Calculation was carried out for the temperature interval $T = 5 \div 12\,\text{MeV}$ using KDE0v1 Skyrme nucleon-nucleon effective interaction [13]. The value of pressure and the temperature interval correspond to subcritical thermodynamic states of nuclear matter. Recall that for the Skyrme force KDE0v1, the values of pressure and temperature at the critical point are about $P_{\text{cr}} = 0.22$ MeV/fm^3 and $T_{\text{cr}} = 14.8$ MeV. The results for A = 48, 120, and 216 are shown in Fig. 1 by the *dash-dotted*, *dashed*, and *dotted lines*, respectively (see Eqs. (4) and (6)). For the purpose of comparison, the calculation at the same pressure and asymmetry parameter was carried out for infinite asymmetric nuclear matter (solid line in Fig. 1, see Eqs. (10) and (13)). Comparing the *dot-dashed*, *dashed*, and *dotted lines* with the solid one in Fig. 1, it is seen that the temperature in the middle of the plateau region for the small systems is lower than that for infinite matter. Also, the results obtained for small systems are smooth and do not demonstrate fracture (derivative discontinuity), which is seen for infinite matter at bubble and dew points shown in Fig. 1 by arrows.

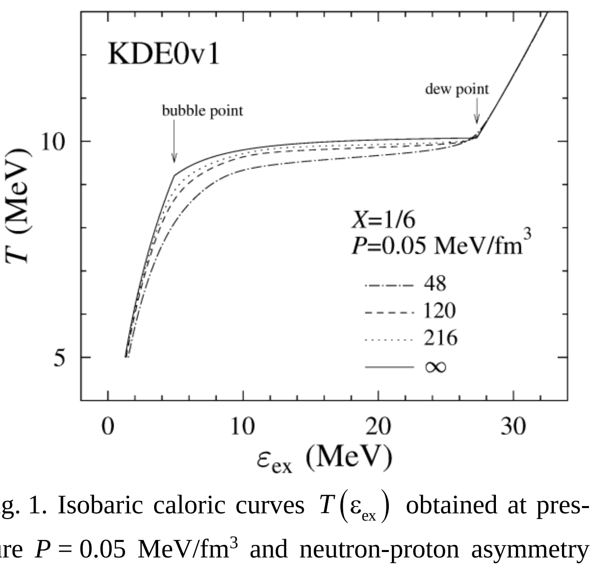

Fig. 1. Isobaric caloric curves $T(\varepsilon_{\text{ex}})$ obtained at pressure P = 0.05 MeV/fm^3 and neutron-proton asymmetry parameter X = 1/6. *Dot-dashed*, *dashed*, and *dotted lines* present the results for small nuclear systems with A = 48, 120, and 216, respectively. The solid line gives the result in the case of infinite matter. The positions of the bubble and dew points of infinite matter are shown by arrows. Calculations were carried out using KDE0v1 Skyrme nucleon-nucleon interaction [13].

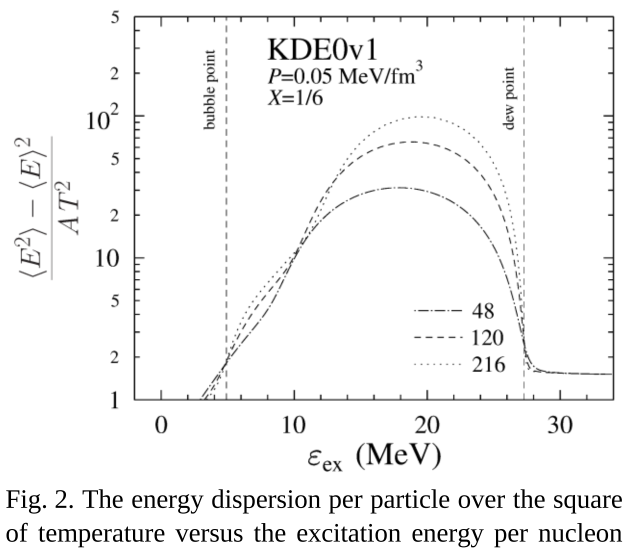

Fig. 2. The energy dispersion per particle over the square of temperature versus the excitation energy per nucleon ε_{ex}, see Eqs. (5) and (6). Results are obtained for small nuclear systems with A = 48 (*dot-dashed line*), 120 (*dashed line*), and 216 (*dotted line*) along the corresponding caloric curves, see Fig. 1. The range of ε_{ex} between bubble and dew points (vertical *dashed lines*) corresponds to the coexistence of liquid and vapor phases for the case of infinite nuclear matter.

We also obtained the energy dispersion $\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2$ by means of Eq. (5). As seen from Eq. (5), the calculation of the dispersion and, consequently, the fluctuation of energy requires the values of second derivatives of the Gibbs thermodynamic potential G with respect to the temperature and pressure. Fig. 2 presents the energy dispersions for small nuclear systems with $A = 48, 120,$ and 216 as functions of excitation energy per particle along the corresponding caloric curves shown in Fig. 1. Fig. 2 demonstrates the increase of energy dispersions in the two-phase region of excitation energies between bubble and dew points. Such an increase, together with the plateau region in the caloric curve $T(\varepsilon_{\text{ex}})$, gives the signature of the occurring phase transition. The presented results for small nuclear systems could be valuable to give an idea of the excitation energy range where to expect the observation of the liquid-vapor phase transition.

Caloric curves illustrated in Fig. 1 are almost flat in the phase coexistence region where they are located within the small temperature interval of about 1 MeV wide. It is of interest to see the behavior of the heat capacity for this quite narrow temperature region. The temperature dependence of the heat capacity per particle at fixed pressure c_P, is displayed in Fig. 3.

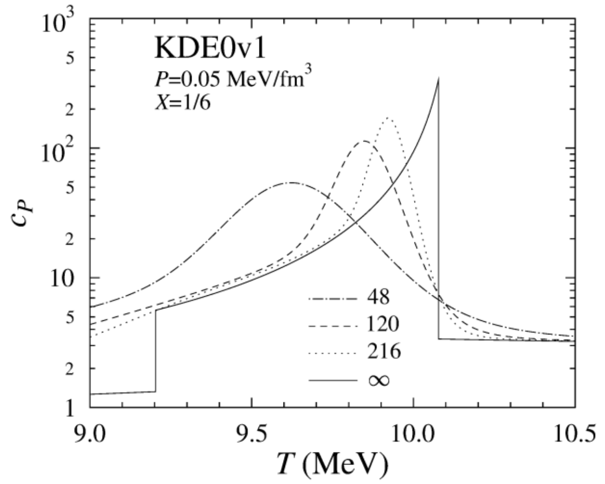

Fig. 3. Heat capacity per particle at fixed pressure as a function of temperature. Presented results correspond to the values of pressure P = 0.05 MeV/fm^3 and asymmetry parameter X = 1/6. The solid line shows the result for infinite nuclear matter. *Dot-dashed*, *dashed*, and *dotted lines* are obtained for small nuclear systems having particle numbers A = 48, 120, and 216, respectively.

Calculations were performed using Skyrme nucleon-nucleon effective interaction KDE0v1 at the same values of pressure and asymmetry parameter as for Figs. 1 and 2. The result of the heat capacity calculation for infinite nuclear matter, see Eq. (15), is presented in Fig. 3 by the solid line. The value of c_P exhibits two abrupt jumps, the one on the left-hand side of the figure corresponds to the temperature of the bubble point, and the other one on the right, which corresponds to the dew point temperature. The behavior of c_P between the bubble and dew point temperatures demonstrates the strong increase in the value of heat capacity. Based on Eq. (7), the values of c_P were obtained for small nuclear systems having numbers of nucleons A = 48 (*dot-dashed* line in Fig. 3), 120 (*dashed line*), and 216 (*dotted line*). As seen from Fig. 3, in contrast to the case of infinite matter, the heat capacity of a small system consisting of a limited number of nucleons is a smooth function of temperature without abrupt jumps. It is peaked at a certain temperature in the considered range of phase coexistence with a very high maximum value (as compared to the ideal gas value of 5/2). The peak temperature and the maximum value increase as the nucleon number of the system goes higher, compare *dot-dashed*, *dashed*, and *dotted lines* in Fig. 3. In a small nuclear system, considering the phase equilibrium of dense particle aggregates (clusters) with the surrounding saturated vapor of nucleons, the existence of a strong peak in the specific heat c_P at subcritical temperature is an evidence of the liquid-vapor phase transition.

4. Summary

Caloric curves $T(\varepsilon_{\text{ex}})$ for small nuclear systems of a limited number of nucleons were considered in comparison with the infinite nuclear matter. The study of small (finite) systems requires the application of statistical mechanics, whereas to describe infinite matter, it is sufficient to apply equilibrium conditions from regular thermodynamics. Calculations of the caloric curves $T(\varepsilon_{\text{ex}})$, energy dispersion $\sigma_E^2(\varepsilon_{\text{ex}})$, and heat capacity per particle $c_P(T)$, were carried out using the KDE0v1 Skyrme-type nucleon-nucleon interaction. As evident from the calculations for infinite matter, see solid lines in Figs. 1 and 3, the transfer between the single phase state and the state of phase coexistence (bubble and dew points) is accompanied by irregularities, like fracture in $T(\varepsilon_{\text{ex}})$ or an abrupt jump in $c_P(T)$. As for the finite system, all the obtained quantities demonstrate smooth behavior. The flat region of the caloric curve, high maximum in heat capacity, and high value of energy fluctuation give us signs of the occurring liquid-vapor phase transition in the finite system.

In this paper, we use a statistical approach considering a small nuclear system (or cluster) as an element of the Gibbs ensemble. The individual cluster can exchange nucleons with other clusters, simulating the intermediate nuclear system at the break-up stage. The mean volume and energy associated with the cluster are totally determined by the statistical weight calculated from the corresponding partition sum. The implementation of this statistical approach is much simpler as compared to the solution of time-dependent equations like, for example, in the quantum molecular dynamic method [14], and gives necessary physical quantities as the ensemble averages. The price of the simplicity is the assumption of thermodynamic equilibrium which still remains the subject of study. The presented results on energy fluctuations for small nuclear systems could be valuable to give an idea of the excitation energy range where to expect the observation of the liquid-vapor phase transition.

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Appendix A

Skyrme nucleon-nucleon effective interaction

In this paper, we adopt the following form of the Skyrme nucleon-nucleon effective interaction $U_{12}\left(\mathbf{r}_1,\mathbf{r}_2;\rho\right)$ [15, 16]:

$$U_{12}\left(\mathbf{r}_1,\mathbf{r}_2;\rho\right)=t_0\left(1+x_0P_{12}^{\boldsymbol{\sigma}}\right)\delta\left(\mathbf{r}_1-\mathbf{r}_2\right)+$$

$$+\frac{1}{2}t_1\left(1+x_1P_{12}^{\boldsymbol{\sigma}}\right)\times\left[\overleftarrow{k}_{12}^2\delta\left(\mathbf{r}_1-\mathbf{r}_2\right)+\delta\left(\mathbf{r}_1-\mathbf{r}_2\right)\vec{k}_{12}^2\right]+$$

$$+t_2\left(1+x_2P_{12}^{\boldsymbol{\sigma}}\right)\overleftarrow{k}_{12}\delta\left(\mathbf{r}_1-\mathbf{r}_2\right)\vec{k}_{12}+$$

$$+\frac{1}{6}t_3\left(1+x_3P_{12}^{\boldsymbol{\sigma}}\right)\rho^{\nu}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2}\right)\delta\left(\mathbf{r}_1-\mathbf{r}_2\right)+$$

$$+iW_0\overleftarrow{k}_{12}\delta\left(\mathbf{r}_1-\mathbf{r}_2\right)\left(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2\right)\times\vec{k}_{12}, \qquad \text{(A1)}$$

where t_i, x_i, ν, and W_0 are parameters of the interaction and $P_{12}^{\boldsymbol{\sigma}}$ is the spin exchange operator, $\boldsymbol{\sigma}_i$ is the Pauli spin operator, $\vec{k}_{12}=-i\left(\vec{\nabla}_1-\vec{\nabla}_2\right)/2$, $\overleftarrow{k}_{12}=-i\left(\overleftarrow{\nabla}_1-\overleftarrow{\nabla}_2\right)/2$. Here, the right and left arrows indicate that momentum operators act on the right (initial state) and on the left (final state), respectively. The nucleon-nucleon interaction (A1) includes central, non-local (momentum-dependent), density-dependent, and spin-orbit terms. The set of parameters for the KDE0v1 Skyrme-type nucleon-nucleon interaction is presented in the Table below.

Parameters of Skyrme nucleon-nucleon interaction KDE0v1 [13]

Parameter	Value (KDE0v1)
t_0(MeV·fm^3)	–2553.0843
t_1(MeV·fm^5)	411.6963
t_2(MeV·fm^5)	–419.8712
t_3(MeV·fm$^{3(1+\nu)}$)	14603.6069
x_0	0.6483
x_1	–0.3472
x_2	–0.9268
x_3	0.9475
ν	0.1673
W_0 (MeV·fm^5)	124.4100

The use of interaction (A1) within the semiclassical Thomas - Fermi theory [12] allows us to express Skyrme - Hartree - Fock energy as a functional of the local neutron and proton densities and build an energy-density functional eliminating the single-particle wave functions. The temperature-dependent Thomas - Fermi approximation, as applied to infinite nuclear matter of uniform particle density, provides the expression for the free energy per particle ϕ_{TF}, namely,

$$\rho\phi_{\text{TF}}\left(\rho,X,T\right)=T\sum_q\left(\eta_q\rho_q-\frac{2}{3}\mathcal{A}_q^*J_{3/2}\left(\eta_q\right)\right)+$$

$$+\frac{1}{2}t_0\left[\left(1+\frac{x_0}{2}\right)\rho^2-\left(x_0+\frac{1}{2}\right)\left(\rho_n^2+\rho_p^2\right)\right]+$$

$$+\frac{1}{12}t_3\rho^{\nu}\left[\left(1+\frac{x_3}{2}\right)\rho^2-\left(x_3+\frac{1}{2}\right)\left(\rho_n^2+\rho_p^2\right)\right], \qquad \text{(A2)}$$

where T is the temperature, q is the isospin index, ρ_q stands for the densities of particle species q ($q = n$ for neutron and $q = p$ for proton), $\rho = \rho_n + \rho_p$ is the total density of nucleons, and $X=\left(\rho_n-\rho_p\right)/\rho$ is the asymmetry parameter. The Fermi integral $J_\beta\left(\eta_q\right)=\int_0^\infty dz\,z^\beta/\left(1+\exp\left(z-\eta_q\right)\right)$ in Eq. (A2) depends on the fugacity η_q. The value of η_q is obtained from the condition

$$\rho_q=\mathcal{A}_q^*J_{1/2}\left(\eta_q\right). \qquad \text{(A3)}$$

Here, $\mathcal{A}_q^*=\frac{1}{2\pi^2}\left(\frac{2m_q^*T}{\hbar^2}\right)^{3/2}$ and m_q^* is the effective nucleon mass derived by

$$\frac{\hbar^2}{2m_q^*}=\frac{\hbar^2}{2m}+\frac{1}{4}\left[t_1\left(1+\frac{x_1}{2}\right)+t_2\left(1+\frac{x_2}{2}\right)\right]\rho+$$

$$+\frac{1}{4}\left[t_2\left(x_2+\frac{1}{2}\right)-t_1\left(x_1+\frac{1}{2}\right)\right]\rho_q\,, \qquad \text{(A4)}$$

where m is the bare nucleon mass. The pressure in the two-phase equilibrium conditions (8) and (9) of Section 3 is associated with the equation of state $P(\rho, X, T)$, which can be obtained from free energy per particle $\phi_{\mathrm{TF}}(\rho, X, T)$ of Eq. (A2) as

$$P(\rho, X, T)=\rho^2\left(\frac{\partial \phi_{\mathrm{TF}}}{\partial \rho}\right)_{X,T}. \qquad \text{(A5)}$$

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ПІДКРИТИЧНІ СТАНИ АСИМЕТРИЧНОЇ ЯДЕРНОЇ МАТЕРІЇ

Розглянуто ізобарну калориметричну криву в області підкритичних станів. Отримано флуктуації енергії вздовж калориметричної кривої для малих ядерних систем що складаються з обмеженої кількості нуклонів. Одержано температурну залежність питомої теплоємності при сталому тиску. Розраховані для малих ядерних систем величини обговорюються та порівнюються з тими ж величинами для нескінченної ядерної матерії.

Ключові слова: калориметрична крива, флуктуація енергії, мала ядерна система.

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