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TUNNELING ALONG γ-AXIS BETWEEN PROLATE AND OBLATE SHAPES

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Using Zickendraht - Dzyublik - Filippov coordinates; we derived equation to determine the rotation and monopole + quadrupole vibrations of the nuclear ellipsoid of inertia. Apart from the monopole part, it coincides with the Bohr - Mottelson equation. However, our mass parameter turns out to be about 2.5 times larger than the hydrodynamic one. The equation is solved quasi-classically for nonrotating β rigid but γ soft nuclei, whose energy landscape has prolate and oblate minima, connected by the collective path along the γ axis. The γ tunnelling strength appears to be twice the usual one, taking place for the one-dimensional potential with two minima, separated by the barrier. The E0 transition strength between levels of the ground 0^+ doublet is calculated. The results are consistent with the experiment for ^{74}Kr.

1. Introduction

The nuclei with neutron and proton numbers $N \approx Z \approx 36$ exhibit mixing of the prolate and oblate shapes. Starting from oblate and prolate trial configurations, Petrovici et al. [1 - 5] performed variational calculations for great number of levels of the nuclei ^{72}Kr, ^{74}Kr and ^{68}Se, which display variety of shapes. The residual interaction is shown to provide mixing of these configurations and repulsion of their energies. Unfortunately, this so-called Excited Vampir approach [1 - 5] completely ignores the β and γ vibrations of the nuclear shape.

More refined adiabatic self-consistent collective coordinate method has been developed in [6 - 10]. As a result of cumbersome time-dependent variational calculations there has been derived the collective one-dimensional equation for large-amplitude vibrations of the nuclear shape, starting from the microscopic Hamiltonian with pairing plus quadrupole interaction. Kobayasi et al. [10] reduced studying of the motion of A nucleons to consideration of the one-dimensional collective motion in the (β, γ) plane. Two minima of the potential energy for the nuclei $N \approx Z \approx 36$ were shown to be connected by a collective path, going along the valley on the potential energy surface in the direction of the γ axis with approximately constant β. In other words, the barrier between the prolate and oblate shapes in the β direction at the point $\gamma = 0$, corresponding to the spherical shape, is much higher than that in the γ direction at $\gamma = \pi/6$, corresponding to maximal triaxiality of the nucleus. Similar statement about shallow potential barrier along the γ axis in ^{74}Kr is presented in [11]. Thus, it is the large-amplitude γ motion that causes mixing of the oblate and prolate shapes in such nuclei.

The calculations [6 - 10] are rather cumbersome. At the same time, much more simple description is ensured by the Bohr - Mottelson equation. However, it has been derived in the framework of the liquid drop model [12, 13]. As a result, the Bohr - Mottelson equation contains the hydrodynamic mass parameter

$$B_{hydr} = \frac{3}{8\pi} AmR_0^2, \qquad (1)$$

where A is the number of nucleons with mass m, and R_0 is the nuclear radius. Unfortunately, B_{hydr} is too small to reproduce experimental data on the moments of inertia.

On the other hand, in papers [14, 15] the kinetic energy operator of A nucleons has been expressed in terms of $3A$ independent variables, which included the Eulerian angles θ_1, θ_2, θ_3 to determine the nuclear rotation and three variables *a, b, c* to describe the size and shape of its inertia ellipsoid. Very similar set of coordinates was previously presented by Zickendraht [16], but it contained 6 redundant variables for description of intrinsic motion. Different algebraic aspects of such approach to the nuclear collective motion have been discussed further in [17 - 20].

Using Zickendraht - Dzyublik - Filippov coordinates [14 - 16], we shall first derive the equation, which determines the nuclear collective motion. Afterwards it will be applied for the quasi-classical analysis of the prolate-oblate mixing of shapes in even-even nuclei. We shall calculate also the electric monopole (E0) transition strength between the levels of the ground 0^+ doublet and compare the results with the experiment [11] for ^{74}Kr.

It is worhtnoting also the paper [21], which analyzed prolate-oblate phase transitions by means of the $O(6)$ group.

2. Collective kinetic energy

The relative motion of A nucleons in the center of mass system is described by the Jacobi vectors $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_{A-1}$. Let us introduce the rotating frame with axes x', y', z' directed along the principal axes of the inertia ellipsoid of the nucleus. Then the projections of $\vec{\xi}_i$ on these axes should satisfy the

following conditions:

$$\sum_{i=1}^{A-1}\xi_{ix'}\xi_{iy'}=\sum_{i=1}^{A-1}\xi_{ix'}\xi_{iz'}=\sum_{i=1}^{A-1}\xi_{iy'}\xi_{iz'}=0\,. \qquad (2)$$

Orientation of the system x', y', z' is determined by the Euler angles θ_1, θ_2, θ_3.

Following [14, 15] we pass to the space of the particle numbers with basis unit vectors $\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_{A-1}$ and introduce there the vectors

$$\vec{A}_{x'}=\sum_{i=1}^{A-1}\xi_{ix'}\vec{e}_i\,,\quad \vec{A}_{y'}=\sum_{i=1}^{A-1}\xi_{iy'}\vec{e}\,,\quad \vec{A}_{z'}=\sum_{i=1}^{A-1}\xi_{iz'}\vec{e}_i\,. \qquad (3)$$

Treating the constraint (2) as an orthogonality condition of these vectors, we introduce vibrational variables as the lengths of the vectors $\vec{A}_{x',y',z'}$ [14 - 16]

$$a=\sqrt{\sum_i\xi_{ix'}^2}\,,\quad b=\sqrt{\sum_i\xi_{iy'}^2}\,,\quad c=\sqrt{\sum_i\xi_{iz'}^2}\,. \qquad (4)$$

In addition the rotating coordinate frame is introduced with unit basis vectors $\vec{e}\,'_1,\vec{e}\,'_2,\cdots,\vec{e}\,'_{A-1}$, whose vectors $\vec{e}\,'_{A-3},\vec{e}\,'_{A-2},\vec{e}\,'_{A-1}$ are directed along $\vec{A}_{x'},\vec{A}_{y'},\vec{A}_{z'}$ respectively, i.e.

$$\vec{A}_{x'}=a\vec{e}_{A-3}\,,\quad \vec{A}_{y'}=b\vec{e}_{A-2}\,,\quad \vec{A}_{z'}=c\vec{e}_{A-1}\,. \qquad (5)$$

The remaining $3A$-9 variables are introduced as the generalized Euler angles [14, 15], which determine the rotation of the coordinate system $\vec{e}\,'_1,\vec{e}\,'_2,\cdots,\vec{e}\,'_{A-1}$ with respect to $\vec{e}_1,\vec{e}_2,\cdots,\vec{e}_{A-1}$.

The kinetic energy of A nucleons, written in such collective coordinates, is [14, 15]

$$T=\frac{m}{2}\Big[\dot{a}^2+\dot{b}^2+\dot{c}^2+a^2\sum_{k\neq A-3}\Omega_{A-3,k}^2+b^2\sum_{k\neq A-2}\Omega_{A-2,k}^2+c^2\sum_{k\neq A-1}\Omega_{A-1,k}^2-$$

$$-4bc\Omega_{A-2,A-1}\omega_{x'}-4ac\Omega_{A-1,A-3}\omega_{y'}-4ab\Omega_{A-3,A-2}\omega_{z'}+ \qquad (6)$$

$$+\left(b^2+c^2\right)\omega_{x'}^2+\left(a^2+c^2\right)\omega_{y'}^2+\left(b^2+a^2\right)\omega_{z'}^2\Big],$$

where $\vec{\omega}=\left\{\omega_{x'},\omega_{y'},\omega_{z'}\right\}$ is the frequency (angular velocity) of the rotation in the real space, and $\Omega_{kk'}$ denotes the frequency of rotation in the plane $\vec{e}_k,\vec{e}_{k'}$ in the abstract A-1 dimensional space.

Let us make the transformation to coordinates ρ,β,γ

$$a=\frac{\rho}{\sqrt{3}}\left[1+\beta\cos\left(\gamma-2\pi/3\right)\right],$$

$$b=\frac{\rho}{\sqrt{3}}\left[1+\beta\cos\left(\gamma+2\pi/3\right)\right], \qquad (7)$$

$$c=\frac{\rho}{\sqrt{3}}\left[1+\beta\cos\gamma\right],$$

Here the hyperradius is given by (see also [22])

$$\rho=\left(a^2+b^2+c^2\right)^{\frac{1}{2}}, \qquad (8)$$

the coordinates β and γ define respectively the deformation and asymmetry (triaxiality) of the inertia ellipsoid. Note, that writing expressions (7) we assumed small deformations, $\beta << 1$. More general transformation, being valid at $\beta \sim 1$, is suggested in [23].

For $\gamma = 0$ the inertia ellipsoid is axially symmetric and prolate ($c > a = b$); for $\gamma = \pi/3$ it is symmetric and oblate ($b < a = c$); the value $\pi/6$ corresponds to its maximal asymmetry. Beyond the interval $0 < \gamma < \pi/3$ we get the repeated nuclear shapes. Therefore for single-valued correspondence of the shapes and coordinates one should demand that the vibrational coordinates vary in the intervals

$$0\le\rho<\infty\,,\quad 0\le\beta<\infty\,,\quad 0\le\gamma\le\pi/3\,. \qquad (9)$$

If there is a uniform distribution of nucleons inside the nucleus with sharp quadrupole surface, then the shape of the nucleus and the inertia ellipsoid coincide. In this case the above parameters β, γ completely coincide with those of Bohr and Mottelson [12, 13].

The vibrational part of the kinetic energy in new coordinates takes the form

$$T_{vibr}\equiv\frac{m}{2}\left(\dot{a}^2+\dot{b}^2+\dot{c}^2\right)=\frac{m}{2}\left(\dot{\rho}^2+\frac{1}{2}\rho^2\dot{\beta}^2+\frac{1}{2}\rho^2\beta^2\dot{\gamma}^2\right). \qquad (10)$$

In the remaining part of T the parameters a, b, c denote now the functions (7).

The Jacobian of the transformation is a product of the collective Jacobian, depending on θ_1, θ_2, θ_3 as

well as on the vibrational coordinate ρ, β, γ, and the Jacobian, associated with the intrinsic rotation in the A-1 dimensional space. The collective Jacobian is given by

$$J_{coll} = \rho^{3A-4}\beta^4 \sin 3\gamma \sin\theta_2 . \qquad (11)$$

Writing down the kinetic energy in terms of the generalized momenta $p_q = \partial T / \partial \dot{q}$ and omitting the terms, responsible for the intrinsic motion and its coupling to the collective motion, one arrives at the following expression for the kinetic energy of the nuclear collective motion

$$T_{coll} = \frac{1}{2m}\left[p_\rho^2 + \frac{2}{\rho^2} p_\beta^2 + \frac{2}{\rho^2\beta^2} p_\gamma^2 \right] +$$

$$+ \frac{1}{4m\rho^2\beta^2}\left[\frac{L_{x'}^2}{\sin^2(\gamma - 2\pi/3)} + \frac{L_{y'}^2}{\sin^2(\gamma + 2\pi/3)} + \frac{L_{z'}^2}{\sin^2\gamma} \right]. \qquad (12)$$

By using standard rules of quantization [24], we get the corresponding operator

$$\hat{T}_{coll} = \hat{T}_\rho - \frac{\hbar^2}{2B(\rho)}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma \frac{\partial}{\partial\gamma} \right] +$$

$$+ \frac{\hbar^2}{8B(\rho)\beta^2}\left[\frac{\hat{I}_{x'}^2}{\sin^2(\gamma - 2\pi/3)} + \frac{\hat{I}_{y'}^2}{\sin^2(\gamma + 2\pi/3)} + \frac{\hat{I}_{z'}^2}{\sin^2\gamma} \right], \qquad (13)$$

where $\vec{L} = \hbar\vec{I}$ is the nuclear spin,

$$\hat{T}_\rho = \frac{\hbar^2}{2m}\frac{1}{\rho^{3A-4}}\frac{\partial}{\partial\rho}\rho^{3A-4}\frac{\partial}{\partial\rho} \qquad (14)$$

is the kinetic energy operator for the monopole vibrations,

$$B(\rho) = \frac{1}{2} m\rho^2 . \qquad (15)$$

When the amplitude of ρ vibrations is small compared to the equilibrium value ρ_0, the mass function B (ρ) can be replaced by the mass parameter

$$B \equiv B(\rho_0) = \frac{1}{2} m\rho_0^2 . \qquad (16)$$

Then the operator $\hat{T}_{coll} - \hat{T}_\rho$ coincides formally with the kinetic energy operator of the Bohr - Mottelson model [12, 13]. But now the mass parameter $B > B_{\text{hydr}}$. Specifically, for an uniform nucleus, confined by a quadrupole shape, defined in the frame x', y', z' by the radius [12, 13]

$$R(\theta',\varphi') = R_0\left[1 + \beta\cos\gamma Y_{2,0}(\theta',\varphi') + \beta\frac{\sin\gamma}{\sqrt{2}}\left(Y_{2,2}(\theta',\varphi') + Y_{2,-2}(\theta',\varphi') \right) \right], \qquad (17)$$

the parameter ρ_0 is related to the radius R_0 by

$$\rho_0^2 = A\left\langle r^2 \right\rangle \approx 0.6R_0^2 , \qquad (18)$$

where $\left\langle r^2 \right\rangle^{1/2}$ is the mean-square radius of the nucleus. Then our mass parameter becomes

$$B = 0.6AmR_0^2 . \qquad (19)$$

Thus,

$$B = 0.8\pi B_{hydr} \approx 2.5 B_{hydr} . \qquad (20)$$

3. γ-soft motion

We assume that the nucleus is rigid with respect to ρ vibrations and consider only the nuclei with spin $I = = 0$. Their collective motion is determined by the equation

$$-\frac{\hbar^2}{2B}\left\{ \frac{1}{\beta^3}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta} \right) + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\left(\sin 3\gamma\frac{\partial}{\partial\gamma} \right) \right\}\Psi(\beta,\gamma) + W(\beta,\gamma)\Psi(\beta,\gamma) = E\Psi(\beta,\gamma) , \qquad (21)$$

where $W(\beta,\gamma)$ denotes a potential energy, depending on the deformation parameters β and γ.

By making standard substitution

$$\Psi(\beta,\gamma) = \beta^{-2}(\sin 3\gamma)^{-1/2}\Phi(\beta,\gamma) , \qquad (22)$$

we remove the first derivatives in Eq. (21):

$$-\frac{\hbar^2}{2B}\left(\frac{\partial^2}{\partial\beta^2}+\frac{1}{\beta^2}\frac{\partial^2}{\partial\gamma^2}\right)\Phi(\beta,\gamma)+W_{eff}(\beta,\gamma)\Phi(\beta,\gamma)=E\Phi(\beta,\gamma)\,, \qquad (23)$$

where the effective potential energy has the form [13]

$$W_{eff}(\beta,\gamma)=W(\beta,\gamma)-\frac{3}{4}\frac{\hbar^2}{B\beta^2}-\frac{9}{8}\frac{\hbar^2}{B\beta^2}\frac{1}{\sin^2 3\gamma}\,. \qquad (24)$$

New wave function $\Phi(\beta,\gamma)$ satisfies the boundary conditions

$$\Phi(\beta,0)=\Phi(\beta,\pi/3)=\Phi(0,\gamma)=0 \qquad (25)$$

and is normalized as

$$\int_0^\infty d\beta\int_0^{\pi/3}d\gamma\left|\Phi(\beta,\gamma)\right|^2=1\,. \qquad (26)$$

Let the potential energy of the nucleus $W(\beta, \gamma)$ have two minima with coordinates $\beta_1 = \beta$ (0), $\gamma = 0$ and $\beta_2 = \beta$ (π/3), $\gamma = \pi/3$, associated with the prolate and oblate shapes, respectively. At every fixed γ the effective potential $W_{eff}(\beta, \gamma)$ reaches local minimum at the point $\beta_0(\gamma)$, depending on γ. When the triaxiality parameter γ varies from 0 to $\pi/3$, the curve $\beta_0(\gamma)$ runs along the bottom of the valley connecting these points. For small β vibrations one can expand $W_{eff}(\beta, \gamma)$ in powers of the displacements β - $\beta_0(\gamma)$

$$W_{eff}(\beta,\gamma)=V(\gamma)+\frac{C(\gamma)}{2}\big(\beta-\beta_0(\gamma)\big)^2+\dots\,, \qquad (27)$$

where

$$V\left(\gamma\right)=W_{eff}(\beta_0(\gamma),\gamma)$$

$$C(\gamma)=\left(\frac{\partial^2 W_{eff}(\beta,\gamma)}{\partial\beta^2}\right)_{\beta=\beta_0(\gamma)}\,. \qquad (28)$$

From now on we shall assume that the stiffness of β vibrations C does not depend on γ. It is interesting to understand when Eq. (23) splits into two independent equations for β and γ motions. First, the nucleus should be rigid with respect to β vibrations, so that its local softness parameter

$$\mu(\gamma)=\beta_{00}/\beta_0(\gamma)<<1\,, \qquad (29)$$

where $\beta_{00}^2/2$ implies the squared amplitude of the ground-state β vibrations with the frequency ω_β:

$$\beta_{00}=\sqrt{\frac{\hbar}{B\omega_\beta}}\,, \qquad \omega_\beta=\sqrt{\frac{C}{B}}\,. \qquad (30)$$

Just condition (29) enables us to replace the factor β^{-2} at $\partial^2/\partial\gamma^2$ in Eq. (23) by $\beta_0^{-2}(\gamma)$.

Another restriction appears when we introduce the coordinates

$$\beta'=\beta-\beta_0(\gamma)\,, \qquad \xi_\beta=\beta'/\beta_{00} \qquad (31)$$

for description of the β vibrations. Such a procedure can be formally regarded as the transformation from the coordinates β, γ to β', and $\gamma' = \gamma$, under which the derivatives transform as

$$\frac{\partial}{\partial\beta}=\frac{\partial}{\partial\beta'}\,, \qquad \frac{\partial}{\partial\gamma}=\frac{\partial}{\partial\gamma'}-\frac{\partial\beta_0(\gamma)}{\partial\gamma}\frac{\partial}{\partial\beta'}\,. \qquad (32)$$

In the following we return to designation γ instead of γ'.

Thus, for separation of the collective motions we should demand also that

$$\left|\frac{\partial\beta_0(\gamma)}{\partial\gamma}\right|<<1\,. \qquad (33)$$

When both conditions (29), (33) are satisfied, Eq. (23) splits into equation for β vibrations

$$\left\{\frac{\hbar\omega_\beta}{2}\left(-\frac{\partial^2}{\partial\xi_\beta^2}+\xi_\beta^2\right)-E_\beta\right\}\chi(\xi_\beta)=0\,, \qquad (34)$$

and another for γ motion

$$\left\{-\frac{\hbar^2}{2M(\gamma)}\frac{\partial^2}{\partial\gamma^2}+V(\gamma)-E_\gamma\right\}\varphi(\gamma)=0\,, \qquad (35)$$

where $M(\gamma)=B\beta_0^2(\gamma)$ is the effective mass varying along the collective path.

As a consequence, the nuclear wave function factorizes:

$$\Phi(\beta,\gamma)=\chi_{n_\beta}(\xi_\beta)\varphi(\gamma)\,. \qquad (36)$$

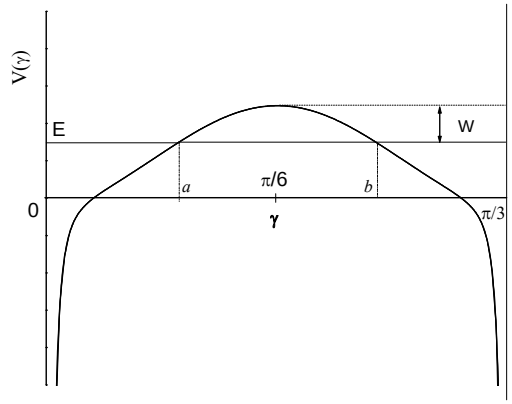
Here $\chi_{n_\beta}(\xi_\beta)$ is an eigenfunction of standard Hamiltonian for the harmonic oscillator with a shifted equilibrium position $\beta_0(\gamma)$; $n_\beta = 0, 1, 2, \dots$ indicates the number of β phonons. The corresponding energy is $E_{n_\beta}=\hbar\omega_\beta(n_\beta+1/2)$. In particular, the ground state of such β oscillator is described by the wave function

$$\psi_0(\xi_\beta) = \frac{1}{\pi^{1/4}\sqrt{\beta_{00}}} \exp\left(-\xi_\beta^2/2\right). \qquad (37)$$

The complete energy of the nucleus will be the sum $E_\beta + E_\gamma$.

4. WKB approximation

Let the energy E_γ be lower than the potential barrier, separating two wells at $\gamma \approx 0$ and $\gamma \approx \pi/3$



Dependence of the symmetric effective potential energy on the triaxial parameter (the classical turning points are denoted by *a* and *b*).

(see the Figure). The energy E_γ equals V_γ in the classical turning points *a* and *b*. In the vicinity of the first minimum β_1,0 the potential energy $W(\beta,\gamma)$ can be expanded in the Taylor series $W(\beta,\gamma) = W(\beta_1,0)+(1/2)W''(\beta_1,0)\gamma^2+....$ Respectively, at $\gamma \approx 0$ the effective potential energy along the collective path is represented by

$$V(\gamma) = \frac{M_1\omega_\gamma^2}{2}\gamma^2 - \frac{\hbar^2}{8M_1\gamma^2}, \qquad (38)$$

where the mass $M_1 = M(0) = B\beta_1^2$. Similarly, at $\gamma \approx \pi/3$

$$V(\gamma) = \frac{M_2\omega_\gamma^2}{2}\left(\gamma - \frac{\pi}{3}\right)^2 - \frac{\hbar^2}{8M_2\left(\gamma - \frac{\pi}{3}\right)^2} + \Delta V_0, \qquad (39)$$

where the mass is $M_2 = M(\pi/3) = B\beta_2^2$ and the energy shift ΔV_0 equals

$$\Delta V_0 = W(\beta_2, \pi/3) - W(\beta_1, 0) + \frac{3\hbar^2}{4M_1} - \frac{3\hbar^2}{4M_2}. \qquad (40)$$

Shrödinger equation (35) with the potential (38) or (39) and the mass $M(\gamma)$ replaced by M_1 or M_2 describes well-known γ vibrations of the axially symmetric prolate or oblate nucleus. The ground states of these γ oscillators are described by the wave functions [13]

$$\varphi_0^{(0)}(\xi_k) = (2/\gamma_{00}^{(k)})^{1/2}\,\xi_k^{1/2}\exp(-\xi_k^2/2), \qquad (41)$$

where k = 1,2 and the dimensionless coordinates are determined by

$$\xi_1 = \gamma/\gamma_{00}^{(1)}, \qquad \xi_2 = (\gamma - \pi/3)/\gamma_{00}^{(2)} \qquad (42)$$

with the oscillator lengths

$$\gamma_{00}^{(1)} = \sqrt{\frac{\hbar}{M_1\omega_\gamma}}, \qquad \gamma_{00}^{(2)} = \sqrt{\frac{\hbar}{M_2\omega_\gamma}}. \qquad (43)$$

Here we assumed the frequencies of γ vibrations in both wells be the same.

At $\gamma \approx 0$ and $\gamma \approx \pi/3$ the main condition of the WKB approximation

$$\left|\frac{\partial\lambda}{\partial\gamma}\right| << 1 \qquad (44)$$

breaks. Here λ is the wavelength, related to the momentum

$$p(\gamma) = \sqrt{2M(\gamma)(E_\gamma - V(\gamma))} \qquad (45)$$

by

$$\lambda(\gamma) = 2\pi\hbar/p(\gamma). \qquad (46)$$

Therefore we ought to use the modified WKB method, developed in [25] in order to solve quasi-classically the radial Shrödinger equation of the hydrogen atom and three-dimensional oscillator. Then we shall omit the divergent part ~ γ^{-2} of the potential (38) and approximate the effective potential V(γ) in the region $0 < \gamma < a$ by $M_1\omega_\gamma^2\gamma^2/2$. An analogous procedure is done for $b < \gamma < \pi/3$, where we replace the potential V(γ) by $\Delta V_0 + M_2\omega_\gamma^2(\gamma - \pi/3)^2/2$.

Since $\varphi(0) = 0$, we choose the wave function at $0 < \gamma < a$ in the form

$$\varphi(\gamma) = \frac{c_1}{\sqrt{p}}\sin\left(\frac{1}{\hbar}\int_0^\gamma p\,d\gamma\right). \qquad (47)$$

Further we use standard matching procedures and introduce as usually the action

$$A = \frac{1}{\hbar}\int_a^b |p|\,d\gamma. \qquad (48)$$

Approximating the barrier ($a < \gamma < b$) by the inverse parabola

$$V(\gamma) \approx -\frac{M(\pi/6)\omega_B^2}{2}\left(\gamma - \frac{\pi}{6}\right)^2 \qquad (49)$$

one finds that

$$A = \frac{\pi W}{\hbar\omega_B}, \qquad (50)$$

where

$$W = V(\pi/6) - E_\gamma \qquad (51)$$

represents the height of the barrier.

Under the barrier the wave function has the form

$$\varphi(\gamma) = \frac{c_1}{\sqrt{|p|}}\left\{\cos\left(\phi_1 - \frac{\pi}{4}\right)e^{A}\exp\left(-\frac{1}{\hbar}\int_\gamma^b |p|d\gamma\right) + \right.$$

$$\left. + \frac{1}{2}\sin\left(\phi_1 - \frac{\pi}{4}\right)e^{-A}\exp\left(\frac{1}{\hbar}\int_\gamma^b |p|d\gamma\right)\right\}, \qquad (52)$$

where we employed the designations

$$\phi_1 = \frac{1}{\hbar}\int_0^a pd\gamma, \qquad \phi_2 = \frac{1}{\hbar}\int_b^{\pi/3} pd\gamma. \qquad (53)$$

Simple calculation gives

$$\phi_1 = \frac{\pi E}{2\hbar\omega_\gamma}, \qquad \phi_2 = \frac{\pi(E - \Delta V_0)}{2\hbar\omega_\gamma}. \qquad (54)$$

From the boundary condition $\varphi(\pi/3) = 0$ one has the following equation

$$4\cot\left(\phi_1 - \frac{\pi}{4}\right)\cot\left(\phi_2 - \frac{\pi}{4}\right) = e^{-2A}. \qquad (55)$$

The wave function in the oblate well takes the form

$$\varphi(\gamma) = \frac{c_2}{\sqrt{p}}\sin\left(\frac{1}{\hbar}\int_{\pi/3}^{\gamma} pd\gamma\right), \qquad (56)$$

where the amplitude

$$c_2 = \frac{1}{2}\frac{\sin(\phi_1 - \pi/4)}{\cos(\phi_2 - \pi/4)}e^{-A}c_1. \qquad (57)$$

5. Energies and wave functions

Let the barrier have small transparency, i.e., $\exp(-2A) << 1$. Then the constraint (55) is fulfilled if

$$\phi_1(E) \approx (n_1 + 3/4)\pi \qquad (58)$$

and/or

$$\phi_2(E) \approx (n_2 + 3/4)\pi \qquad (59)$$

where $n_i = 0, 1, 2, \ldots$. In the absence of tunneling the nucleus is located in one of the potential wells, and the energy levels are determined by exact Bohr-Sommerfeld equalities (58), (59). Using expressions (54) for ϕ_i, one has the energies

$$\epsilon_1 = (2n_1 + 3/2)\hbar\omega_\gamma,$$
$$\epsilon_2 = \Delta V_0 + (2n_2 + 3/2)\hbar\omega_\gamma. \qquad (60)$$

These formulas up to the common unessential constant $\hbar\omega_\gamma/2$ correctly reproduce energy levels the two-dimensional oscillator with the angular momentum $l = 0$ and major oscillator quantum number $N = 2n + l$.

In the case of close-lying levels $\epsilon_1 \approx \epsilon_2$, expanding $\phi_k(E)$ in E - ϵ_k and keeping only linear terms, we transform Eq. (55) to

$$x^2 - \Delta x - \upsilon^2 = 0, \qquad (61)$$

where the following notations are introduced:

$$x = E - \epsilon_1, \quad \Delta = \epsilon_1 - \epsilon_2, \quad \upsilon = \frac{\hbar\omega_\gamma}{\pi}e^{-A}. \qquad (62)$$

The parameter υ determines the tunneling strength through the barrier. Note that $\upsilon = 2\upsilon_0$, where υ_0 is the tunneling strehgth in the case of the potential $V(x)$, having two minima and tending to $+\infty$ as $x \to \pm\infty$.

From Eq. (61) we get the physical (perturbed) energies (see also [27, 28]):

$$E^{\pm} = \frac{1}{2}(\epsilon_1 + \epsilon_2) \pm \frac{1}{2}\sqrt{\Delta^2 + 4\upsilon^2}. \qquad (63)$$

As usually, the energy of the ground state 0_1^+ is taken to be zero ($E^- \equiv E_{0_1^+} = 0$). Then the energy of the first excited 0_2^+ state equals

$$E^+ \equiv E_{0_2^+} = \sqrt{\Delta^2 + 4\upsilon^2}. \qquad (64)$$

The wave functions of this ground 0^+ doublet are

$$\varphi_\pm(\gamma) = c_1^\pm\varphi_0^{(0)}(\xi_1) + c_2^\pm\varphi_0^{(0)}(\xi_2). \qquad (65)$$

Using Eq. (57), one finds the ratio of their amplitudes $R^\pm = c_2^\pm / c_1^\pm$

$$R_{\pm} = -\frac{2\upsilon}{\Delta \pm \sqrt{\Delta^2 + 4\upsilon^2}}. \qquad (66)$$

The wave functions $\varphi_+(\gamma)$ and $\varphi_-(\gamma)$ are orthogonal, since $R_+ R_- = -1$.

The complete wave functions for the ground 0^+ doublet take then the form

$$\Phi_{0_1^+}(\beta,\gamma) = \left(1+R_-^2\right)^{-1/2}\left[\Phi_1^{(0)}(\beta,\gamma) + R_-\Phi_2^{(0)}(\beta,\gamma)\right],$$

$$\Phi_{0_2^+}(\beta,\gamma) = \left(1+R_+^2\right)^{-1/2}\left[\Phi_1^{(0)}(\beta,\gamma) + R_+\Phi_2^{(0)}(\beta,\gamma)\right], \qquad (67)$$

where the prolate-oblate constituents are

$$\Phi_{1(2)}^{(0)}(\beta,\gamma) = \chi_0(\xi_\beta)\varphi_0^{(0)}(\xi_{1(2)}). \qquad (68)$$

When $\epsilon_1 = \epsilon_2$, the lower level 0_1^+ is described by the symmetric superposition of the functions $\Phi_1^{(0)}$, $\Phi_2^{(0)}$ and the upper level 0_2^+ by the antisymmetric one.

6. Electric monopole transitions

Now we shall consider the electric-monopole transitions between the levels 0_2^+ and 0_1^+. For the nucleus like an uniformly charged quadrupole drop the E0 transition operator reads [12, 13]

$$M(E0) = \frac{3Z}{4\pi}\left(\beta^2 + \frac{5\sqrt{5}}{21\sqrt{\pi}}\beta^3\cos 3\gamma\right), \qquad (69)$$

where Z is the nuclear charge number. For the E0 transition strength between the initial state 0_2^+ and final 0_1^+

$$B\left(E0; 0_2^+ \to 0_1^+\right) = \left|M_{fi}(E0)\right|^2. \qquad (70)$$

one finds the expression

$$B\left(E0; 0_2^+ \to 0_1^+\right) = qB_{\max}\left(E0; 0_2^+ \to 0_1^+\right), \qquad (71)$$

where

$$q = \frac{4}{\left(1+R_+^2\right)\left(1+R_-^2\right)}, \qquad (72)$$

and $B_{\max}(E0)$ stands for the maximal value of $B(E0)$ being achieved in the case of complete mixing of shapes

$$B_{\max}\left(E0; 0_2^+ \to 0_1^+\right) = \left(\frac{3Z}{8\pi}\right)^2\left\{\beta_1^2 - \beta_2^2 + \frac{5\sqrt{5}}{21\sqrt{\pi}}\left[\beta_1^3 + \beta_2^3 + \frac{3}{2}(\beta_1+\beta_2)\left(\frac{\hbar}{B\omega_\beta} - \frac{3}{2}\frac{\hbar}{B\omega_\gamma}\right)\right]\right\}^2. \qquad (73)$$

The factor (72) can be rewritten as

$$q = 4c^2(1-c^2) = 4\left(\upsilon / E_{0_2^+}\right)^2 \qquad (74)$$

where $c^2 = \left(c_2^-\right)^2$ is the weight of the oblate shape in the ground state 0_1^+.

The measured E0 transition strength in ^{74}Kr between the first excited 0_2^+ state with the energy $E_{0_2^+} = 0.509$ Mev and the ground 0_1^+ state lies in the limits $0.13 < B_{\exp}(E0; 0_2^+ \to 0_1^+) < 0.18$ [11]. Following [11], we take $\beta_1 = 0.4$, $\beta_2 = 0.2$, $\hbar\omega_\gamma = 1.689$ Mev and $\hbar\omega_\beta \approx 1.5$ Mev. The mass parameter B is calculated by means of Eq. (19), where the radius is determined by familiar expression

$$R_0 = 1.3A^{1/3}\text{fm}. \qquad (75)$$

The calculated transition strength coincides with the experimental one, if $10\,\% < c^2 < 15\,\%$ and $0.15 \text{ Mev} < \upsilon < 0.18$ Mev. Note that Petrovici et al. [5] predicted for ^{74}Kr the weight $c^2 = 30\,\%$ or $47\,\%$ depending on the chosen fitting parameters. At the same time, they predicted $B(E0; 0_2^+ \to 0_1^+) = 0.03$. Using Eq. (64) we found in addition the difference of the unperturbed 0^+ energy levels: $0.35 \text{ Mev} < |\Delta| < 0.41$ Mev, that is much larger than the estimation $\Delta = -0.071$ Mev given in [5].

Discussion

We derived Eqs. (13), (14) for the collective kinetic energy operator $\hat{T}_{coll}$ of the nuclear ellipsoid of inertia. Apart from the monopole part $\hat{T}_\rho$ the operator $\hat{T}_{coll}$ coincides with that, derived by Bohr and Mottelson [12, 13] for the drop of the irrotational liquid. However, the mass parameter (19) appears now to be equal approximately to 2.5 B_{hydr}.

For the nuclei with large-amplitude γ vibrations one can not separate the rotation from vibrational motion. Therefore we limited ourselves by

consideration of the nuclei with spin $I = 0$. In general case the corresponding equation (21) describes intermixed β and γ motions. We have shown that it reduces to one-dimensional equation for γ motion (35), when the nucleus is rigid with respect to β vibrations and the curve $\beta_0(\gamma)$, drawing the collective path between prolate and oblate shapes, has small slope to the γ axis.

The Schrödinger equation (35) for γ motion is solved by means of the modified WKB method [25]. There the centrifugal barrier is omitted in the radial Schrödinger equations for the hydrogen atom and 3-dimensional oscillator. In analogy we omit the divergent term of the effective potential energy of the same origin as the centrifugal barrier.

The derived equation (61) for the physical energies has the same form as in the case of one-dimensional symmetric potential $V(x)$, which has two minima and grows monotonically as $x \to \infty$ [27, 28]. But now the γ tunneling strength doubles, that has simple quasi-classical explanation. If at the initial moment $t = 0$ the particle is located in one of the potential wells of $V(x)$, then at $t \geq 0$ there arise periodic transitions (Rabi oscillations) from one well to another with the Rabi frequency $\Omega_R = \upsilon_0 / \hbar = (\hbar\omega_0 / 2\pi)e^{-A}$ (see, e.g., the discussion [30] of the molecule NH_3). Here the factor e^{-A} denotes the transmission probability through half a barrier from the turning point $x = a$ to the frontier between the well $x = (a + b)/2$ (the transmission probability through the whole barrier equals e^{-2A}, if $2A >> 1$ [25 - 27]). The second frequency factor $\omega_0 / 2\pi$ represents the knocking rate, which indicates how many times per second a classical particle knocks the barrier at the turning point $x = a$, which lies on one side of the potential well.

On the other hand, the γ motion is restricted within the limits from 0 to $\pi/3$. At $\gamma \approx 0$ there are transverse γ vibrations of the prolate nuclear shape in the plane x', y' perpendicular to the symmetry axis z'. These vibrations can be described as the radial motion of the particle inside the two-dimensional parabolic potential in analogy to the deformation vibrations of the CO_2 molecule [31]. Such classical particle, swinging in the well, reaches the turning points on both sides of the well (the radial coordinate $\gamma = a$) two times during the period of vibrations. Then the knocking rate doubles (compared to that for vibrations in the one-dimensional potential $V(x)$), i.e., taking the value ω_γ/π. The same situation exists at $\gamma \approx \pi/3$. Hence, in the case of the large-amplitude γ motion the tunneling strength $\upsilon = 2\upsilon_0$ (here we put $\omega_0 = \omega_\gamma$).

Using experimental data of Korten [11], we came to the conclusion that the unperturbed vibrational levels in two potential wells of ^{74}Kr are not degenerate, and mixing of the oblate and prolate shapes of this nucleus, although significant, are not complete, as it was assumed previously in [11]. The derived Eqs. (71) - (73) completely reproduce experimental values of E0 transition strengths. Note that the β and γ vibrations give few percent corrections to $B_{\max}$(E0) (for rigid nucleus with $\omega_{\beta(\gamma)} \to \infty$ the maximal E0 transition strength equals 0.37). So the enhancement of these transitions observed in [11] is caused by the prolate-oblate mixing.

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ТУНЕЛЮВАННЯ ВЗДОВЖ ОСІ γ МІЖ ВИТЯГНУТОЮ ТА СПЛЮСНУТОЮ ФОРМАМИ

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Використовуючи координати Зікендрата - Дзюблика - Філіпова, ми отримали рівняння, що визначає обертання монопольно-квадрупольних коливань еліпсоїда інерції ядра. За винятком монопольної частини, воно збігається з рівнянням Бора - Моттельсона. Але наш масовий параметр виявляється у 2,5 рази більший, ніж гідродинамічний масовий параметр. Рівняння розв'язується квазікласично для ядер із жорстким β, але м'яким γ, поверхневі енергії яких мають два мінімуми, що відповідають витягнутій та сплюснутій формам ядра та зв'язані з колективною доріжкою вздовж осі γ. Інтенсивність γ-тунелювання виявляється у два рази більшою, ніж звичайна для одновимірного потенціалу, який має витягнутий та сплюснутий мінімуми, відокремлені бар'єром. Разраховано інтенсивність E0 переходів основного 0^+ дублета. Результати погоджуються з експерементом для ^{74}Kr.

ТУННЕЛИРОВАНИЕ ВДОЛЬ ОСИ γ МЕЖДУ ВЫТЯНУТОЙ И СПЛЮСНУТОЙ ФОРМАМИ

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Используя координаты Зикендрата - Дзюблика - Филиппова, мы получили уравнение, которое определяет вращение монопольно-квадрупольных колебаний эллипсоида инерции ядра. За исключением монопольной части, оно совпадает с уравнением Бора - Моттельсона. Однако наш массовый параметр получается в 2,5 раза больше, чем гидродинамический массовый параметр. Уравнение решается квазиклассически для ядер с жестким β, но мягким γ, чьи поверхностные энергии имеют вытянутый и сплюснутый минимумы, связанные с коллективной дорожкой вдоль γ оси. Интенсивность γ-туннелирования получается в два раза больше, чем обычная для одномерного потенциала, имеющего два минимума, разделеные барьером. Рассчитаны силы переходов E0 основного 0^+ дублета. Результаты согласуются с эксперементом для ^{74}Kr.

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