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MULTISCALE SIMULATION ALGORITHM FOR STOCHASTIC COOLING SIMULATION

The subsequent steps on improvement of the numerical methods for solution of the stochastic cooling simulation are described. The algorithmic approach for improvement of solution of one dimensional Fokker - Planck equation (FPE) is presented in the paper. This approach is based on a multiscale simulation algorithm to increase the accuracy of the FPE solution.

Keywords: stochastic cooling, Fokker - Planck equation, momentum spread, antiproton beam.

Introduction

Investigation of the particle motion under the influence of noise is very important problem of accelerator physics. For the correct using of accelerators one should know the following: what is the long time behavior of the dynamics, what is the probability for the particle to hit the vacuum chamber (and then be lost) (mean first passage time), what are the average fluctuations of the particle around the periodic design orbit of the accelerator (moments), and what is the time evolution of the probability density (transient and stationary behavior). In order to answer on these questions the stochastic cooling system is used.

In our previous papers, the main principles of stochastic cooling were described [1 - 3]. For the theoretical investigations the space fractional FPE with instantaneous source is considered and different numerical schemes for solving FPE are used [4 - 6]. It was shown that using the different methods of fractional derivatives the FPE is transformed into a system of ordinary differential equations. Numerical results for FPE with a constant diffusion coefficient are evaluated for analyses of the stochastic cooling.

In this paper the numerical solution of FPE using a time and space scaling during numerical solving of FPE is described. It will be shown that scaling of FPE is able to increase the accuracy of FPE solution.

Time scaling simulation algorithm

The FPE describes the change of Probability Density Function (PDF) or $\psi(z, t)$ in a space z and time t , where z can be interpreted either as a momentum spread $\Delta p/p$ or as an emittance ϵ . For calculations of stochastic cooling processes, the general FPE for the PDF ($\psi(z, t)$) evolution of one space variable z at time t has the form

$$\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial z} \left(F\psi - \frac{1}{2} D \frac{\partial \psi}{\partial z} \right), \quad (1)$$

where $F(z, t)$ and $D(z, t)$ are known functions which may depend, in principle, on time, and $\psi(z, t)$ represents the unknown solution.

Stochastic momentum cooling is operated to obtain a high-density beam within a small momentum spread for experiments. The goal of the cooling is reduction of momentum spread and emittance in all directions: transverse and longitudinal. As result: the phase space density of particles will be increased. A FPE is used as a powerful tool for investigating the stochastic momentum cooling process. We study the momentum spread of the particles distribution of which is described by function $\psi = \psi(z, t)$, where $z = \Delta p/p$ is momentum spread. In the case of stochastic cooling F describes the cooling force and D is diffusion term. These coefficients characterize the given stochastic cooling system.

We are interested in the time evolution of the density ψ . Such evolution looks like as it is presented by Fig. 1. During the cooling, the peak of the distribution function becomes sharper [4].

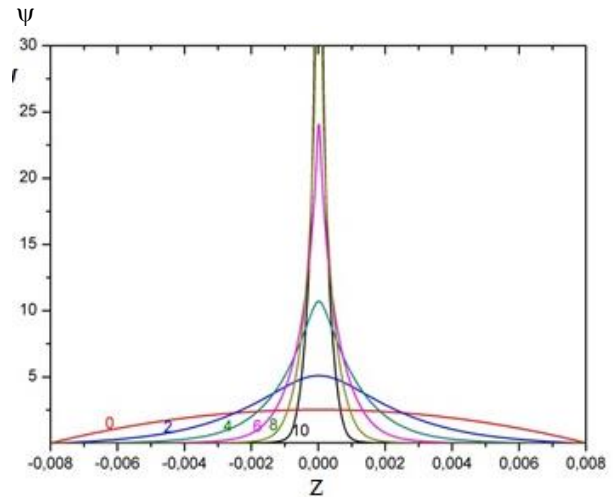


Fig. 1. Evolution of the PDF during the stochastic cooling.

In ref. [4 - 6] several numerical methods were developed for simulation of FPE. It has been shown that stochastic cooling is modeled by the Fokker - Planck, where the evolution of the PDF is studied. It was shown that using the different methods of fractional derivatives the FPE is transformed into a system of partial differential equations (PDE). Then the PDE system can be solved by different methods [4, 5].

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The solution of FPE is based on that the PDE (partial differential equation) algorithm computes values ψ which approximate the true solution ψ on the grid of certain size (N_i, N_j) by $\psi_{ij} \psi(i\Delta z, j\Delta t)$ for $i = 0, 1, 2, \dots$ and $j = 0, 1, \dots, N$. Using such discretization and finite difference method a set of ordinary linear equations can be build, which in matrix form for the fixed i can be written as

$$(I - \Delta t \theta L)\psi_{j+1} = (I - \Delta t(1 - \theta)L)\psi_j, \quad (2)$$

where I is the $N \times N$ identity matrix. L is the tri-diagonal matrix obtained from the differential operator, θ is the parameter (see Ref. [4]). The choice of the parameter θ affects the stability of the method. Common choices are $\theta = 0$ (Euler scheme), $\theta = 1/2$ (Crank Nicolson scheme) or $\theta = 1$ (implicit Euler scheme).

In our own implementation the Crank Nicolson method is used. The solution of FPE is stable depending of the choice of Δt .

The accuracy of this approximation depends on the step sizes Δt and Δz . The smaller these values are the more accurate is the approximation (and the more expensive is the algorithm). The algorithm works by considering a grid row with fixed t at a time, starting with an approximation of the initial condition: $(\psi_{00}, \dots, \psi_{N0})$. Then, in each step, the algorithm uses the approximation $(\psi_{0j}, \dots, \psi_{Nj})$ for time $j\Delta t$ to compute an approximation for time $(j + 1)\Delta t$. The dimension of the domain of the equation is assumed that PDF has a distribution, which is closed to a normal distribution with a small variance. One sees that the PDEs of FPE type are discretized in space by a finite volume method. The PDEs preserve the total probability and the non-negativity of the solutions and the numerical method shares these properties. The expected values are also non-negative in the numerical solution. If the solutions are allowed to be negative, then severe numerical instabilities occur. The solution is advanced in time by an unconditionally stable implicit method and thus problems with stiff systems are avoided. If the space discretization is stable, then there is a unique and positive steady state solution.

Important progress has been made towards efficient algorithms for discrete FPE simulation. The leaping forward method is proposed which can take time step smaller than the initial time step for a solution of FPE. The leaping method shows a promising direction to improve the accuracy of FPE solution. But it has a limitation, in that they are based on the assumption that the probability function $\psi(z, t)$, should not change significantly during each time step. This requires the population of the species to be large relative to one.

The computational work increases slowly with the dimension of the problem compared to the exponential growth with a deterministic method for the master equation. A disadvantage of the method is that the time steps between the PDF can be very small compared to the prevailing dynamics of the system if there are separate time scales in the solution. The fast scale requires small time increments but for an accurate solution it is sufficient to follow the slow scale.

It is proposed to use the next time step scaling simulation algorithm:

Given initial time t , initial state Δt and total cooling time T_{cool} .

Step 1. Compute the PDF state for the initial time t .

Step 2. For $j = 1, \dots, N_s$, calculate a $\psi(z, t)$.

Step 3. FPE is solved using Eq. (2), $\psi(z, t + \Delta t)$ is calculated having $\psi(z, t)$.

Step 4. The time step for the next FPE iteration is given by $\Delta t = \Delta t - dt$, where dt is calculated by tau-leaping formula

$$dt = \tau \frac{\Delta t^2}{T_{cool}} \frac{1}{F}, \quad (3)$$

where the τ parameter defines the leaping rate, F is the decreasing factor of Δt .

Step 5. If $t > T_{cool}$, stop. Otherwise update the time $t = t + \Delta t$.

Go to Step 2.

Numerical simulations for the antiproton beam cooling

The method given above is applied for solutions of FPE, which is described in Ref. [1]. Fig. 2 shows the evolution of the momentum spread during cooling of antiprotons at the Collector Ring (CR) of the FAIR complex (Darmstadt, Germany). In the simulation the $T_{cool} = 20$ s, the initial time step $\Delta t = 10^{-4}$, $\tau = 50$, $F = 100$. This means the time step must be changed from 10^{-4} to 10^{-6} as it is shown in Fig. 3.

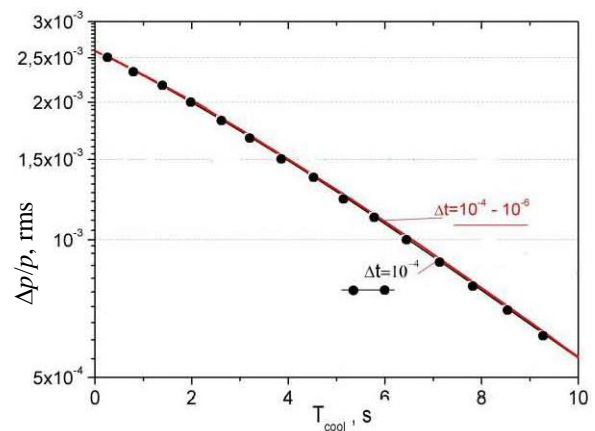


Fig. 2. Momentum spread evolution due to stochastic cooling calculated by 1D FPE. The time step is 10^{-4} s and time step is varied from 10^{-4} to 10^{-6} s.

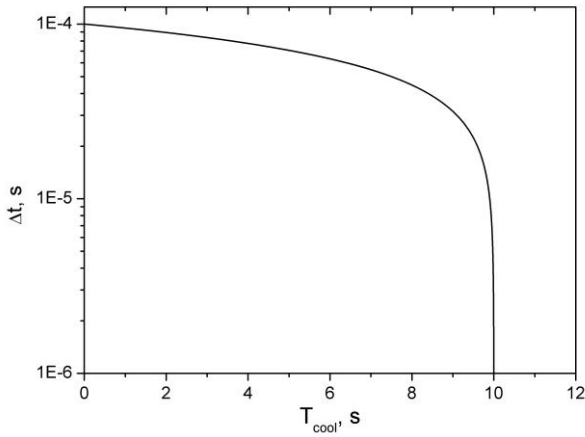


Fig. 3. Changing of the time step in the FPE calculated by Eq. (3).

Fig. 2 and Fig. 3 confirm that the momentum spread evolution as well as the accuracy of the proposed method for the changed time step represents the result obtained with constant time step quite well.

Grid step scaling

This approach has the advantage over other methods that it often scales linearly with the number of discrete nodes used. The solving FPE means calculation of the PDF (or ψ) function over a certain time as shown in Fig. 4. It is assumed that the FPE is solved approximately (with a given accuracy) on a grid i with a given grid points density N_i . When the rms value of the PDF becomes rather small and comparable with a grid step ΔZ (see Fig. 4, *b*) obviously the accuracy of the solutions is very low. Furthermore a solution of the FPE can be obtained on any grid N_k with a given effort from a solution on a coarser grid N_i . In other words, the grid is narrowed as shown in Fig. 5, *a*. The ρ ratio of the neighbo-

ring grid steps is assumed to be not constant, while the grid number remains constant $N_k = N_i$. One can write that for the new grid step ΔX

$$\rho = \frac{\Delta X}{\Delta Z} < 1 \tag{4}$$

is the ratio of grid points on “neighboring” grids and it is assumed to be constant throughout the grid hierarchy, and ΔZ is a grid step of the coarser grid and ΔX is a grid step of a new denser grid N_k as shown in Fig. 5, *b*. This means the PDF is calculated on the grid, which has a variable grid step. In Fig. 5, *a* the PDF, which has been calculated at time t_k and has narrow broad width, is shown on the new dense grid.

Algorithm for simulation of grid step scaling is the next.

1. A 2D two-scale mesh is constructed with a given grid step ΔZ .
2. The domain Ω into a coarse mesh $J(\Omega)$ is decomposed with a certain number of grids N_i .
3. After certain evolution time of the ψ on the coarse mesh $J(\Omega)$ the ψ function is saved.
4. The new domain χ in a new mesh $K(\chi)$ is decomposed with the number of grids $N_k = N_i$.
5. The finite element space over the composite fine mesh is defined. New step grids ΔX is calculated.
6. Then each coarse block $[I, I + 1]$ is subdivided into subgrid blocks as shown in Fig. 5, *b*, where coarse space $J(\Omega)$ is defined on $K(\chi)$ and subgrid space ΔX is defined for each coarse block ΔZ .
7. The ψ function is defined on the fine mesh $K(\chi)$. Unknown points X_k at nodes of fine mesh are calculated through known four points Z_i using interpolation Langleage algorithm (see Fig. 5, *b*).

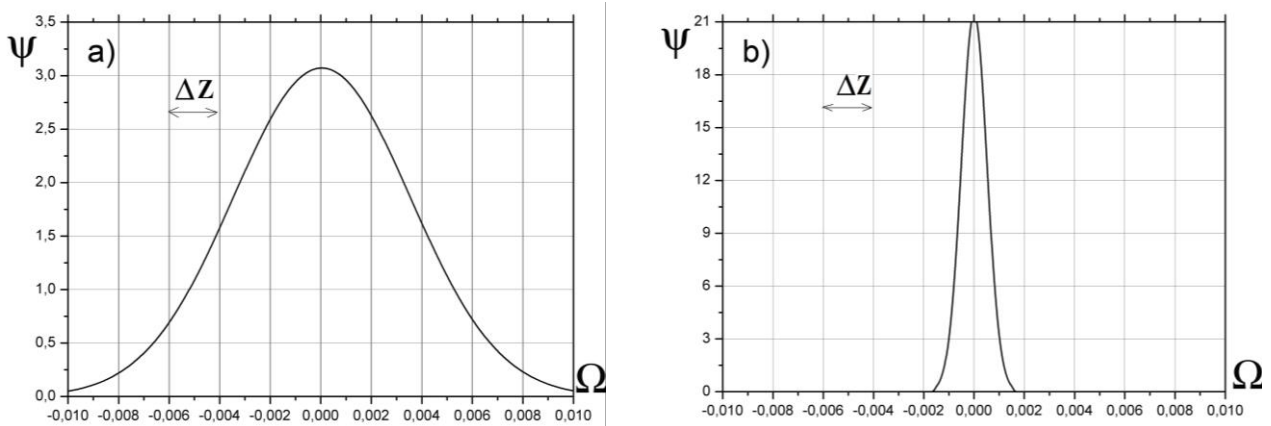


Fig. 4. ψ function evolution on the static grid mesh (*a* – for initial time, *b* – later time).

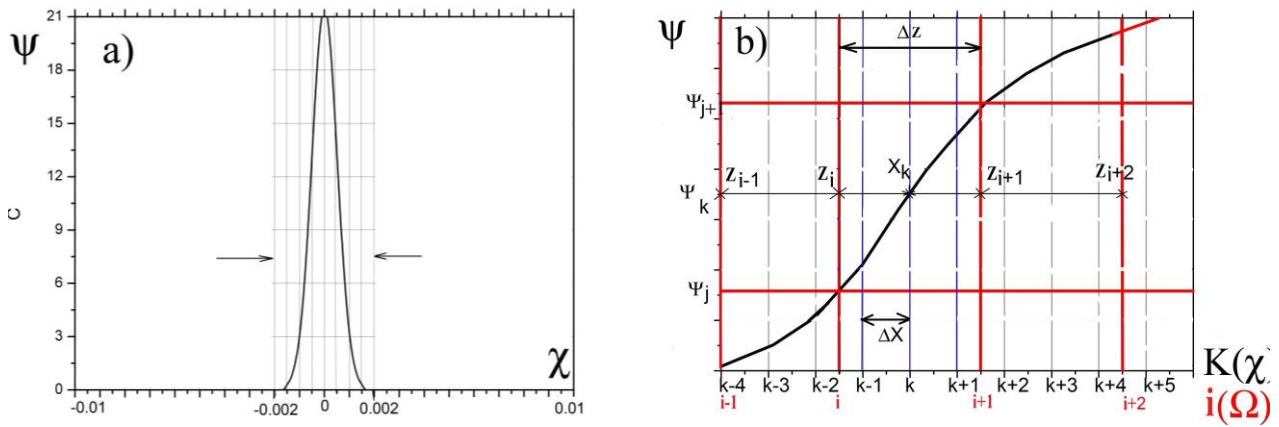


Fig. 5. Grid narrowing (a) and psi function calculation on the new narrowed grid (b).

Numerical simulations for the antiproton cooling

In order to demonstrate the efficiency and accuracy of the proposed method for solution of FPE, the numerical simulation of antiproton cooling at the CR has been done. The parameters of the CR cooling system used in simulation are given in Ref. [4 - 7].

Fig. 6, a shows the evolution of the momentum spread in two cases: when the grid is stable and grid is variable, where the grid transformations are performed as written above. Fig. 6, b shows the grid step variations applied in this test simulation.

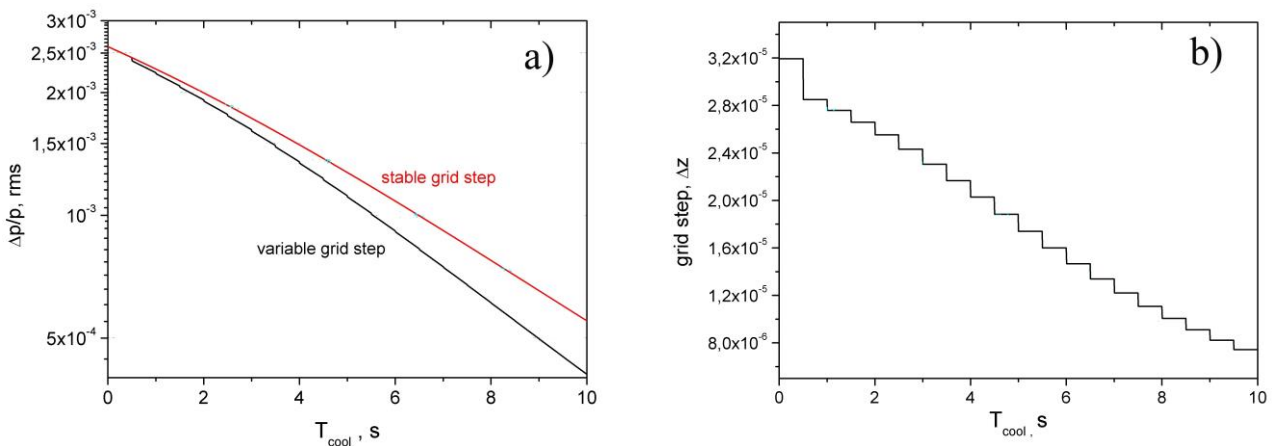


Fig. 6. Momentum spread evolution calculated for antiproton beam cooling at the CR (a); grid step narrowing during psi function evolution (b).

Conclusions and futures

In this paper, the fast and precise method for the higher dimensions stochastic cooling calculations has been discussed. The special code has been developed to study the beam dynamic in storage ring, where the stochastic cooling process must be used.

Presented method can be used for the wide range of tasks and will be applied to more realistic parameters for studying the dynamics of particles in storage rings.

In the near future, this work will be extended to investigation of stochastic cooling systems, which can allow to cool beam by different methods: TOF, Notch filter and Palmer [1].

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ЧИСЕЛЬНИЙ АЛГОРИТМ БАГАТОКРАТНОГО МАСШТАБУВАННЯ ДЛЯ СИМУЛЯЦІЙ СТОХАСТИЧНОГО ОХОЛОДЖЕННЯ

Описано подальші кроки щодо вдосконалення чисельної методики розрахунків процесів стохастичного охолодження. Наведено удосконалений алгоритм вирішення одномірного рівняння Фоккера - Планка. Цей підхід базується на симуляційному алгоритмі багатократного масштабування, що дає змогу збільшити точність рішення розв'язку рівняння Фоккера - Планка.

Ключові слова: стохастичне охолодження, рівняння Фоккера - Планка, імпульсний розкид, антипротонний пучок.

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ЧИСЛЕННИЙ АЛГОРИТМ МНОГОКРАТНОГО МАСШТАБІРОВАНИЯ ДЛЯ СИМУЛЯЦИИ СТОХАСТИЧЕСКОГО ОХЛАЖДЕНИЯ

Описаны последующие шаги по совершенствованию численной методики расчетов процессов стохастического охлаждения. Данная статья описывает усовершенствованный алгоритм решения одномерного уравнения Фоккера - Планка. Этот подход базируется на симуляционном алгоритме многократного масштабирования, который позволяет увеличить точность решения уравнения Фоккера - Планка.

Ключевые слова: стохастическое охлаждение, уравнение Фоккера - Планка, импульсный разброс, антипротонный пучок.

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