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PARTICLE TUNNELING AND SCATTERING IN A THREE-DIMENSIONAL POTENTIAL WITH A HARD CORE AND AN EXTERNAL POTENTIAL BARRIER

V. S. Olkhovsky, M. V. Romaniuk

Institute for Nuclear Research, National Academy of Sciences of Ukraine, Kyiv

The non-relativistic particle tunneling and scattering through a spherical three-dimensional potential barrier (either rectangular, or Coulomb repulsion), containing a spherical potential rectangular well with a hard core inside, had been studied. The explicit analytical expressions for the S-matrix of elastic scattering and all probability amplitudes (external and internal reflections, tunneling inside and tunneling outside) for zero angular momentum and for relations between them had been firstly obtained. In conclusion, unlike to the typical simplified one-dimensional approximation, utilized for low-energy astrophysical nuclear-fusion reactions, we underline the necessity to consider the three-dimensional picture which brings to the multiple internal reflections from internal barrier wall and also to the more strict penetration factor.

Keywords: three-dimensional tunneling, scattering, barrier, potential well, hard core.

Introduction

The time-dependent aspects in the process of the quantum tunneling of particles through 1-dimensional (1D) barriers had been the object of several papers, reported, for instance, in [1] and [2] (and in many references quoted therein). The main results of [1, 2] consist in the self-consistent definitions of tunneling and reflection durations for 1D transitions. However, up to now the time analysis of the three-dimensional (3D) tunneling had not been practically presented. Really, particle tunneling through a 3D barrier had been studied only in a simplified stationary way in the framework of the WKB approximation (see, for example [3 - 6]), or using only the elementary time-dependent description in applications for some concrete tasks such as α-decay.

We study the non-relativistic particle tunneling and scattering through a spherical three-dimensional potential barrier (either rectangular, or Coulomb-repulsion). In the central part of the system we shall also consider a spherical potential rectangular well with a hard core. Our approach is based on the generalization of the paper [7] where only a rectangular well and a rectangular barrier had been considered without a hard core, without a Coulomb repulsion barrier and also out of the framework of the WKB.

Our consideration will be limited here by the stationary wave functions and zero orbital quantum number ($l = 0$). We will describe the impact of the particles, also following [7], as a sequence of two successive processes: in the first stage an ingoing wave packet tunnels through the barrier inside the well, producing a reflected wave in the external region; in the second phase, an outgoing wave from the well (which has been appeared after passing of the ingoing tunneling wave packet) tunnels through the barrier and produces, finally, an outgoing mode in the external region and a reflected wave in the internal region. Then, we describe the sub-barrier low-energy scattering as a whole for the considered potential picture, with obtaining the *S*-matrix and the relation between the *S*-matrix and all the probability amplitudes. Concluding, we indicate to the importance of taking into account the multiple internal reflections from the internal barrier wall and also the strict quantum-mechanical penetration factor in the scattering and nuclear-fusion *S*-matrix, unlike to the typical 1D approximation utilized in low-energy sub-barrier astrophysical nuclear collisions (see, for instance, [8 - 12]).

Impact from outside

The schemes of the impact from outside for the case of the rectangular and Coulomb-repulsion barriers are shown in Fig. 1. Following [7] for the case of a rectangular well (but now with a hard core) and a rectangular barrier, we shall refer to the various region in this way. Region I with $r > R_2$ represents the external region of null potential; region II delimited by R_1 and R_2 is the barrier region; region III with $R_0 < r < R_1$ is the well; region IV with $r < R_0$ is the infinite hard core with null wave function. In the case of the Coulomb barrier the vertical line $r = R_2$ separates the external above-barrier region I (where the particle kinetic energy E is larger than the height of the Coulomb-repulsion barrier) and the internal under-barrier region II (where E is less than the height of the Coulomb-repulsion barrier); the vertical line $r = R_1$ separates the well III and the maximal height of the Coulomb-repulsion barrier.

The radial stationary wave function for the potential schemes in Fig. 1, *a* and *b*, which satisfies the radial Schroedinger equation, will be

$$\begin{aligned} &\phi_I^{(ex)} = e^{-ik_1\rho} + A_R^{(ex)} e^{ik_1\rho} \qquad \rho = r - R_0 \qquad R_2 \le r < \infty \\ &\phi_{II}^{(ex)} = \alpha_1 e^{-\chi\rho} + \beta_1 e^{\chi\rho} \qquad R_1 \le r < R_2 \\ &\phi_{III}^{(ex)} = A_T^{in} e^{-ik_2\rho} \qquad R_0 < r < R_1 \qquad (1) \end{aligned}$$

(for the scheme 1, *a*) and

$$\Phi_I^{(ex)} = [G_0(k_1,\eta,r) - iF_0(k_1,\eta,r)] +$$

$$+A_R^{ex}[G_0(k_1,\eta,r) + iF_0(k_1,\eta,r)], \qquad R_1 \le r < \infty,$$

$\Phi_{II}^{(ex)}$ is formally the same as $\Phi_I^{(ex)}$, $R_1 \le r < R_2$,

$$\Phi_{III}^{(ex)} = A_T^{in} e^{-ik_2\rho}, \quad \rho = r - R_0, \qquad R_0 < r < R_1 \qquad (2)$$

(for the scheme1, *b*), respectively. Here k_1 and $E = \hbar^2 k_1^2/2m$ are the wave number and the kinetic energy, $\chi = [2m(V_1 - E)]^{(1/2)}/\hbar$, $k_2 = [2m(V_0 + E)]^{(1/2)}/\hbar$. For the Coulomb- repulsion barrier

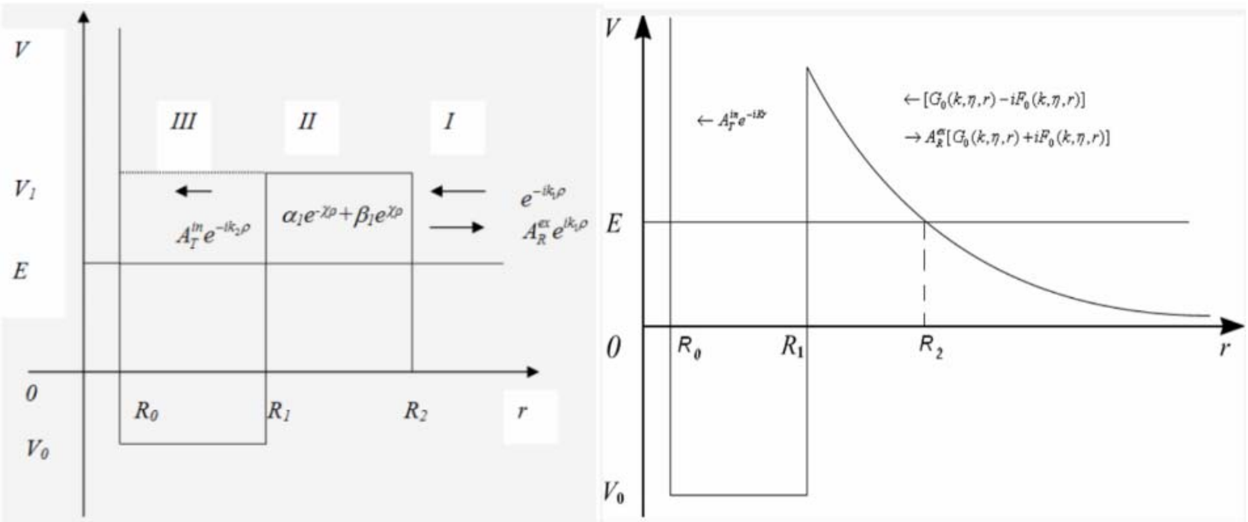
$$V_c = \frac{z_1 z_2 e^2}{r}, \qquad R_1 \le r < \infty \qquad (3)$$

$z_1 e$ and $z_2 e$ are the particle charges. The Coulomb functions $G_0(k_1, \eta, r)$ and $F_0(k_1, \eta, r)$ have the asymptotic

$$G_0(k_1,\eta,r) \xrightarrow[r\to\infty]{} \cos(k_1 r - \eta \ln 2k_1 r + \sigma),$$

$$F_0(k_1,\eta,r) \xrightarrow[r\to\infty]{} \sin(k_1 r - \eta \ln 2k_1 r + \sigma), \qquad (4)$$

$\eta = \dfrac{z_1 z_2 e^2 m}{\hbar^2 k_1}$ is the Sommerfeld parameter, $\sigma = \arg \Gamma(1 + i\eta)$.



a *b*

Fig. 1. Schematic description of the impact of the ingoing wave with the rectangular (*a*) or Coulomb-repulsion barrier from outside (*b*).

In addition, we have that $A_R^{(ex)}$, α_1, β_1 and $A_T^{(in)}$ are, respectively, the external reflection amplitude factor, the evanescent and anti-evanescent wave amplitude factors during the first tunneling and the internal transmission amplitude factor. Their analytical expression can be found by imposing the continuity condition for both the stationary wave functions and their first derivatives at the points $r = R_2$ and $r = R_1$, finding for the external reflection coefficient the expression:

$$A_R^{ex} = \frac{[\exp(-2ik_1(R_2 - R_0))(\exp(2\chi(R_2 - R_1))](\chi + ik_1)(\chi - ik_2) - (\chi - ik_1)(\chi + ik_2)}{-[\exp(2\chi(R_2 - R_1))](\chi - ik_1)(\chi - ik_2) + (\chi + ik_1)(\chi + ik_2)}, \qquad (5)$$

which, in the limit $\chi(R_2 - R_1)$ tending to infinity, becomes $-\exp(-2ik_1(R_2 - R_0))\dfrac{(\chi + ik_1)}{(\chi - ik_1)}$. The internal transmission (tunneling) amplitude A_T^{in} is equal to

$$A_T^{in} = \frac{4i\chi k_1 \exp((-\chi + ik_2)(R_1 - R_0) + (\chi - ik_1)(R_2 - R_0))}{-[\exp(2\chi(R_2 - R_1))](\chi - ik_1)(\chi - ik_2) + (\chi + ik_1)(\chi + ik_2)}, \qquad (6)$$

which, in the same limit as before, tends to 0.

As in [7], one obtains

$$\left|A_R^{ex}\right|^2 + \frac{k_2}{k_1}\left|A_T^{in}\right|^2 = 1 \qquad (7)$$

being the consequence of the conservation law for the probability fluxes.

For the case (2) the external reflection amplitude A_R^{ex} is equal to

$$A_R^{ex} = -\frac{[G_0(k,\eta,R_1) - iF_0(k,\eta,R_1)]ik_2 + [G_0'(k,\eta,R_1) - iF_0'(k,\eta,R_1)]k_1}{[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]ik_2 + [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k_1} \tag{8}$$

the internal transmission (tunneling) amplitude A_T^{in} is equal to

$$A_T^{in} = \frac{2ik_1 \exp(ik_2(R_1 - R_0))}{[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]ik_2 + [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k_1}, \tag{9}$$

and also equation (7) is valid if the known relation $F_0 G_0' - G_0 F_0' = 1$ for the wronskian is taken into account. Here G_0' and F_0' are the derivatives of G_0 and F_0 with respect to k_1R_1, respectively.

For very small k_1 when $k_1 \to 0$ (more precisely, when $2\eta >> k_1R_1$)

$$G_0 \to 2\left(\frac{k_1R_1}{\pi}\right)^{\frac{1}{2}} I_0\left(2\left(2\pi k_1R_1\right)^{\frac{1}{2}}\right)\exp(\pi\eta),$$

with $I_0\left(2\left(2\pi k_1R_1\right)^{\frac{1}{2}}\right) \to 1$,

$$G_0' \to -2\left(\frac{2\eta}{\pi}\right)^{\frac{1}{2}} K_0\left(2\left(2\pi k_1R_1\right)^{\frac{1}{2}}\right)\exp(\pi\eta),$$

with $K_0\left(2\left(2\pi k_1R_1\right)^{\frac{1}{2}}\right) \to \ln\left(1/\gamma(2\pi k_1R_1)^{\frac{1}{2}}\right)$,

$$F_0 \to \left(\pi k_1R_1\right)^{\frac{1}{2}} I_1\left(2\left(2\pi k_1R_1\right)^{\frac{1}{2}}\right)\exp(-\pi\eta),$$

with $I_1\left(2\left(2\pi k_1R_1\right)^{\frac{1}{2}}\right) \to \left(2\pi k_1R_1\right)^{\frac{1}{2}}$,

$$F_0' \to \left(2\pi k_1R_1\right)^{\frac{1}{2}} I_0\left(2\left(2\pi k_1R_1\right)^{\frac{1}{2}}\right)\exp(-\pi\eta),$$

γ = 1.781… being the Euler constant, and if $(2k_1^2/k_2^2\eta/k_1R_1)[\ln\gamma^{-1}(2\pi k_1R_1)^{-1/2}] << 1$, the transmission (penetration) probability from outside through the coulomb barrier into the internal rectangular potential well $\left|A_T^{in}\right|^2$ becomes

$$\left|A_T^{in}\right|^2 \to \frac{\pi k_1}{k_2^2 R_1}\exp(-2\pi\eta). \tag{10}$$

So, in distinction from 1D WKB approximation which is typical for low-energy analysis, in the 3D case one has even for very small k_1 to take into account not only the exponential factor $\exp(-2\pi\eta)$, but also the pre-exponential factor $\frac{\pi k_1}{k_2^2 R_1}$.

For the rectangular barrier (see Fig. 1, *a*) the phase times of the tunnelling through the barrier inwards from outside and of the external reflection form the barrier outside, $\tau_T^{ph(in)}$ and $\tau_R^{ph(ex)}$, can be defined as the evident generalization of the one-dimensional (1D) definitions:

$$\tau_T^{ph(in)} = \hbar\frac{\partial((\arg A_T^{(in)}(E)e^{-ik_2R_1}))}{\partial E} +$$

$$+\frac{R_2}{v_1} \xrightarrow[\chi(R_2-R_1)]{} \left(\frac{1}{v_1}+\frac{1}{v_2}\right)/\chi, \quad v_{1,2} = \frac{\hbar k_{1,2}}{m} \tag{11}$$

and

$$\tau_R^{ph(ex)} = \hbar\frac{\partial(\arg(A_R^{(ex)}(E)e^{-ik_1R_2}))}{\partial E} +$$

$$+\frac{R_2}{v_1} \xrightarrow[\chi(R_2-R_1)]{} \frac{2}{v_1\chi}. \tag{12}$$

So we can see from expressions (8) and (9), for analogy with the similar one-dimensional quantities, the manifestation of the Hartmann effect (i.e. the absence of the dependence of the phase and mean tunneling time on the barrier width R_2 - R_1 for sufficiently large R_2 - R_1, or more precisely $\chi(R_2 - R_1) >> 1$).

Emission from (out of) the barrier

The schemes of the emission from the barrier for the case of the rectangular and Coulomb- repulsion barriers are shown in Fig. 2.

The radial stationary wave functions for the schemes in Fig. 2 will be

$$\begin{aligned} \phi_{III}^{in} &= e^{ik_2\rho} + A_R^{in}e^{-ik_2\rho}, && R_0 < r < R_1, \\ \phi_{II}^{in} &= \alpha_2 e^{-\chi\rho} + \beta_2 e^{\chi\rho}, && R_1 \le r < R_2, \\ \phi_I^{in} &= A_T^{ex} e^{ik_1\rho}, && R_2 \le r < \infty \end{aligned} \tag{13}$$

and

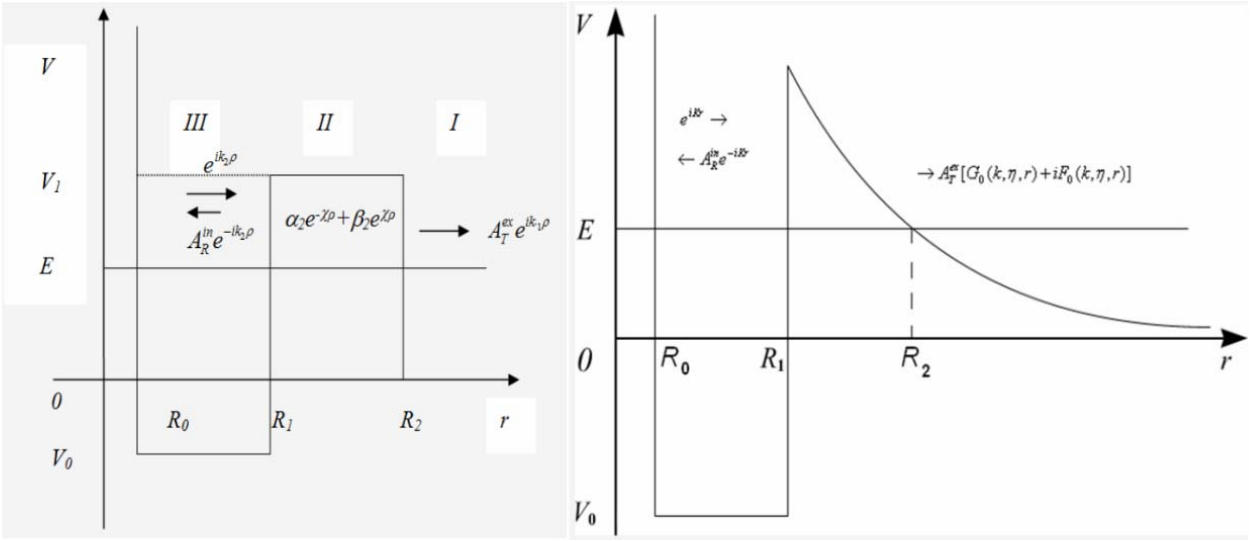
$$\Phi_{III}^{(in)} = e^{ik_2\rho} + A_R^{in} e^{-ik_2\rho}, \quad R_0 < r < R_1,$$

$\Phi_{II}^{(ex)}$ is formally the same as $\Phi_I^{(in)}$, ($R_1 \le r < R_2$),

$$\Phi_I^{(ex)} = A_T^{ex}[G_0(k,\eta,r) + iF_0(k,\eta,r)], \ \rho = r - R_0,$$

$$R_1 \le r < \infty \qquad (14)$$

respectively.



a *b*

Fig. 2. Schematic description of the emission of the internal outgoing wave through the rectangular (*a*) and Coulomb-repulsion barrier (*b*).

Imposing the continuity conditions for the stationary wave functions and their derivatives at the point $r = R_2$ and $r = R_1$, we find the analytical expressions for the amplitudes A_R^{in}, α_2, β_2 and A_T^{ex}. In particular, the internal reflection amplitude A_R^{in} is equal to

$$A_R^{in} = \frac{[\exp(2ik_2(R_1 - R_0))(\exp(2\chi(R_2 - R_1))](\chi - ik_1)(\chi + ik_2) - (\chi + ik_1)(\chi - ik_2)}{-[\exp(2\chi(R_2 - R_1))](\chi - ik_1)(\chi - ik_2) + (\chi + ik_1)(\chi + ik_2)}. \qquad (15)$$

The external transmission (tunneling) amplitude A_T^{ex} is equal to

$$A_T^{ex} = \frac{4i\chi k_2 \exp((\chi + ik_2)(R_1 - R_0)) + (\chi - ik_1)(R_2 - R_0))}{-[\exp(2\chi(R_2 - R_1))](\chi - ik_1)(\chi - ik_2) + (\chi + ik_1)(\chi + ik_2)}. \qquad (16)$$

For the case (2) the internal reflection amplitude A_R^{in} is equal to

$$A_R^{in} = e^{2ik_2(R_1 - R_0)} \frac{[G_0(k_1,\eta,R_1) + iF_0(k_1,\eta,R_1)]ik_2 - [G_0'(k_1,\eta,R_1) + iF_0'(k_1,\eta,R_1)]k_1}{[G_0(k_1,\eta,R_1) + iF_0(k_1,\eta,R_1)]ik_2 + [G_0'(k_1,\eta,R_1) + iF_0'(k_1,\eta,R_1)]k_1} \qquad (17)$$

and

$$A_T^{ex} = \frac{2ik_2 \exp(ik_2(R_1 - R_0))}{[G_0(k_1,\eta,R_1) + iF_0(k_1,\eta,R_1)]ik_2 + [G_0'(k_1,\eta,R_1) + iF_0'(k_1,\eta,R_1)]k_1}. \qquad (18)$$

And also as in [7], one obtains

$$\left|A_R^{in}\right|^2 + \frac{k_1}{k_2}\left|A_T^{ex}\right|^2 = 1 \qquad (19)$$

as a consequence of the conservation law for the probability fluxes.

Repeating for (18) the same reasoning, as we made for very small k_1 when $k_1 \to 0$ after (9) till (10), we obtain

$$\left|A_T^{ex}\right|^2 \to \frac{\pi}{k_2 R_1}\exp(-2\pi\eta). \qquad (20)$$

So also for (20), in distinction from 1D WKB approximation which is typical for low-energy analysis, in the 3D case one has to take into account for very small k_1 not only the exponential factor $\exp(-2\pi\eta)$, but also the pre-exponential factor $\frac{\pi}{k_2 R_1}$.

For the rectangular barrier (see Fig. 2, *a*) the phase times of the internal reflection from the inward barrier wall inside the well and of the tunneling through the barrier outwards from the internal barrier wall, $\tau_R^{Ph(in)}$ and $\tau_T^{Ph(ex)}$, can be also defined by the evident generalization of the general 1D definition from [1, 2]:

$$\tau_R^{Ph(in)} = \hbar \frac{\partial(\arg(A_R^{(in)}(E) e^{-ik_2 R_1}))}{\partial E} -$$

$$-\frac{R_1}{v_2} \xrightarrow[\chi(R_2 - R_1)]{} \frac{2}{\chi v_2} \qquad (21)$$

and

$$\tau_T^{Ph(ex)} = \hbar \frac{\partial(\arg(A_T^{(ex)}(E) e^{-ik_2 R_2}))}{\partial E} -$$

$$-\frac{R_1}{v_2} = \hbar \frac{\partial(\arg(A_T^{(ex)}(E))}{\partial E} \equiv$$

$$\equiv \tau_T^{Ph(in)} \xrightarrow[\chi(R_2 - R_1)]{} \left(\frac{1}{v_1} + \frac{1}{v_2}\right) / \chi . \qquad (22)$$

So, we see from (21) - (22) again the manifestation of the Hartman effect with the absence of the dependence on R_2 - R_1.

Scattering matrix

Now we shall connect two mechanisms of scattering described above in one single scattering event, introducing (as in [7]) the *S*-matrix of scattering and considering the multiple reflections inside the potential well. For this purpose we describe the radial stationary wave function for potential schemes in Fig. 3 as:

$$\psi_I = e^{-ik_1\rho} - \tilde{S} e^{ik_1\rho}, \quad \rho = r - R_0, \quad R_2 \le r < \infty,$$

$$\psi_{II} = \alpha\, e^{-\chi\rho} + \beta\, e^{\chi\rho}, \qquad R_1 \le r < R_2,$$

$$\psi_{III} = A(e^{-ik_2\rho} - e^{ik_2\rho}), \qquad R_0 < r < R_1,$$

$$\psi_{IV} \equiv 0, \qquad r \le R_0, \qquad (23)$$

where $\tilde{S} = S e^{2ik_1 R_0}$, and

$$\Psi_I = [G_0(k_1,\eta,r) - iF_0(k_1,\eta,r)] -$$

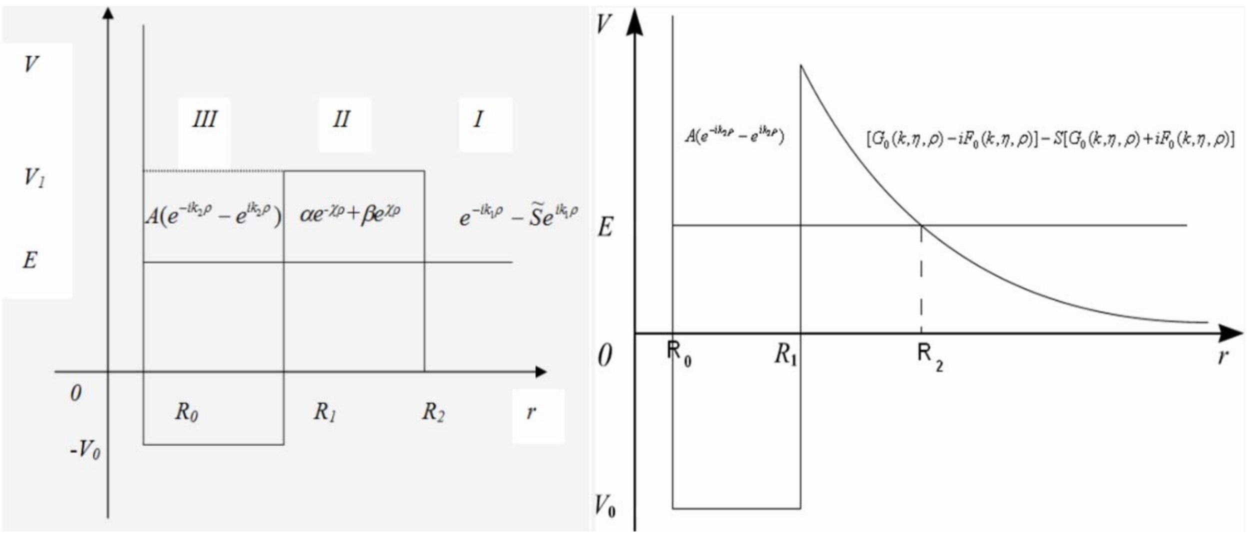
$$-S[G_0(k_1,\eta,r) + iF_0(k_1,\eta,r)], \qquad r \ge R_2,$$

Ψ_{II} is the same as Ψ_I for $\qquad R_1 \le r < R_2$,

$$\Psi_{III} = A(e^{-ik_2\rho} - e^{ik_2\rho}), \qquad R_0 < r < R_1,$$

$$\Psi_{IV} \equiv 0, \qquad r \le R_0 \qquad (24)$$

respectively.



a *b*

Fig. 3. Schematic stationary description of the scattering as a whole for the rectangular (*a*) and Coulomb-repulsion barrier (*b*).

Imposing the continuity conditions for the stationary wave functions and their derivatives at the point $r = R_2$ and $r = R_1$, we find the analytical expressions for the S-matrix, the amplitude A and the amplitudes α and β. In particular, for the case (23) the analytical expressions for $\tilde{S}$ and A are:

$$\tilde{S} = \frac{e^{-2ik_1(R_2-R_0)}(\chi+ik_1)(ik_2-\chi)+(\chi+ik_2)e^{2ik_2(R_1-R_0)})}{(\chi-ik_1)(ik_2-\chi)+(ik_2+\chi)e^{2ik_2(R_1-R_0)})+e^{-2\chi(R_2-R_1)}(\chi+ik_1)(ik_2+\chi)+(-\chi+ik_2)e^{2ik_2(R_1-R_0)})} +$$

$$+\frac{e^{-2ik_1(R_2-R_0)}e^{-2\chi(R_2-R_1)}(ik_1-\chi)(\chi+ik_2)+(-\chi+ik_2)e^{2ik_2(R_1-R_0)})}{(\chi-ik_1)(ik_2-\chi)+(ik_2+\chi)e^{2ik_2(R_1-R_0)})+e^{-2\chi(R_2-R_1)}(\chi+ik_1)(ik_2+\chi)+(-\chi+ik_2)e^{2ik_2(R_1-R_0)})},$$

$$A = \frac{4ik_1\chi\exp\{(\chi+ik_2)(R_1-R_0)-(\chi+ik_1)(R_2-R_0)\}}{(ik_1-\chi)(ik_2-\chi)+(ik_2+\chi)e^{2ik_2(R_1-R_0)})+e^{-2\chi(R_2-R_1)}(\chi+ik_1)(-ik_2-\chi)+(\chi-ik_2)e^{2ik_2(R_1-R_0)})}, \quad (25)$$

And for the case (24) the analytical expressions for S and A are:

$$S = \frac{[G_0(k_1,\eta,R_1)-iF_0(k_1,\eta,R_1)]k_2\cos k_2(R_1-R_0)-[G_0'(k_1,\eta,R_1)-iF_0'(k_1,\eta,R_1)]k_1\sin k_2(R_1-R_9)}{[G_0(k_1,\eta,R_1)+iF_0(k_1,\eta,R_1)]k_2\cos k_2(R_1-R_0)-[G_0'(k_1,\eta,R_1)+iF_0'(k_1,\eta,R_1)]k_1\sin k_2(R_1-R_0)},$$

$$A = \frac{[2ie^{ik_2(R_2-R_0)}k_1]}{[G_0'(k_1,\eta,R_1)+iF_0'(k_1,\eta,R_1)]k_1(1-e^{2ik_2(R_1-R_0)})+[G_0(k_1,\eta,R_1)+iF_0(k_1,\eta,R_1)]ik_2(1+e^{2ik_2(R_1-R_0)})}. \quad (26)$$

As in [7], we can see that $|S| = |\tilde{S}| = 1$; and can be convinced by the direct comparison of A and S with A_R^{ex}, A_T^{ex}, A_R^{in} and A_T^{in} with the proper calculations (derivations) that

$$A = \frac{A_T^{in}}{1+A_R^{in}} \quad (27)$$

and

$$S = -A_R^{ex} + AA_T^{ex} = -A_R^{ex} + \frac{A_T^{ex}A_T^{in}}{1+A_R^{in}} \quad (28)$$

(formulas (27) and (28) are valid for the rectangular and Coulomb-repulsion barrier, respectively).

Considering the expressions (10) and (20) repeating for A_R^{ex} and A_R^{in} the same reasoning's, as we made after (9) till (10) in the case $k_1 \to 0$, we obtain

$$A_R^{ex} \underset{k_1\to 0}{\to} -1, \quad (29)$$

$$A_R^{in} \underset{k_1\to 0}{\to} e^{2ik_2(R_1-R_0)}, \quad (30)$$

$$S \underset{k_1\to 0}{\to} 1, \quad (31)$$

$$T = 1 - S \underset{k_1\to 0}{\to} 0,$$

up to the terms of order $\exp(-2\pi\eta)$.

It is interesting that for a rectangular barrier all the amplitudes A_R^{ex}, A_T^{ex}, A_R^{in} and A_T^{in} depend on the hard-core radius R_0 through the factors $R_1 - R_0$ and $R_2 - R_0$. For a Coulomb-repulsion barrier only A_T^{in}, A_R^{in} and A_T^{ex} depend on R_0 through the factor $e^{[ik_2(R_1-R_0)]}$ which is important inside a Coulomb-repulsion barrier. This dependence is a consequence of the null values of radial wave functions inside a hard core.

In [7] the multiple internal reflections in (27) an (28) are present even without the internal hard core as a consequence of the infinite radial internal reflections between the diameter-opposite points of the internal spherical 3D barrier wall, similar to the internal multiple reflections between two 1D barriers. The physical meaning of the term $\frac{1}{1+A_R^{in}}$ is directly connected to the presence of a infinite sequence of coherent multiple internal reflections that can be described by the stationary wave functions:

$$A_T^{(in)}(1-A_R^{(in)}+(A_R^{(in)})^2-(A_R^{(in)})^3+...)e^{-ikr} = \frac{A_T^{(in)}}{1+A_R^{(in)}}e^{-ikr}, \quad (32)$$

$$A_T^{(in)}(1-A_R^{(in)}+(A_R^{(in)})^2-(A_R^{(in)})^3+...)e^{ikr} = \frac{A_T^{(in)}}{1+A_R^{(in)}}e^{ikr}, \quad (33)$$

for the ingoing and the outgoing waves respectively.

Now, with the presence of the hard core in the interval (0, R_0), we have an infinite series of coherent radial internal reflections between the hard core and the diameter-opposite points of the internal spherical 3D barrier wall (see Fig. 3).

Such coherent multiple internal reflections do usually take place when the bombarding charged particles are elementary (protons, positrons, π^--mesons etc). And they do also take place for non-resonance and resonance scattering.

But if a bombarding charged particle is a cluster, like, for instance, the alpha-particle, and we have a resonance scattering with the formation of the alpha-radioactive compound nucleus, then we have a more complicated *non-coherent* process: the alpha-particle is vanishing inside the compound (parent) nucleus almost at the nuclear surface and then, after a certain virtual and real sojourn time τ_{soj} (including the motion under the surface before the vanishing and after the new formation), the alpha-particle appears again near the nuclear surface and penetrates through the coulomb barrier outwards (see, for instance, papers [14, 15]).

Assuming that, after every exit "portion" of the α-particle probabilistic wave packet in the infinite series of multiple coherent internal reflections with the successive its impacts and tunneling exits outwards, the probability of every successive impact onto the internal wall of the Coulomb-repulsion barrier is decreasing by the factor $|A_R^{in}|^2$ incomparison with the preceding impact, we can represent the total probability of the α-decay (evidently equal to 1, if we recall in advance (19)) as an infinite sum of the decreasing geometrical progression:

$$(k_1/k_2)|\left|A_T^{ex}\right|^2|^2[1+|A_R^{in}|^2+|A_R^{in}|^2|A_R^{un}|^2+\ldots]=$$

$$=(k_1/k_2)\left|A_T^{ex}\right|^2/[1-|A_R^{in}|^2]=1. \qquad (34)$$

Of course, we suppose here (in the total accordance with the above-mentioned indication on the α-particle vanishing and formation inside parent nucleus near the barrier) that various steps of multiple internal reflections, after the α-particle formation near the parent-nucleus surface till its successive vanishing (dissolving) inside the parent nucleus, are *incoherent* relative each other because of the independence of the successive vanishing processes and hence not the probability amplitudes but the probabilities have be summed over the succession of multiple internal reflections.

Further, we do naturally presume in [16] that during every step of the α-particle incoherent multiple internal reflections the general mean duration τ_{soj} of the α-particle virtual and real existence inside the parent nucleus, after its previous internal reflection till its successive internal reflection, is equal to the sum of the mean time of the α-particle vanishing and successive formation processes, the mean time of its reflection time inside the nucleus and the mean time of its kinematical motion after reflection inside and towards the surface after formation. And the quantity τ_{soj} is the same for any pair of successive α-particle impacts.

The effective, or mean, value of the *alpha-particle virtual and real sojourn time inside the parent alpha-radioactive nucleus between the successive incoherent multiple internal reflections* during the long alpha-decay had been evaluated in a phenomenological way, basing on the simple base of the experimental *exponential law* of the α-decay with the mean time τ [16]:

$$L(t)=\exp(-t/\tau). \qquad (35)$$

If $\tau_{soj} << \tau$, then during τ_{soj} the decay probability decreases by the quantity

$$\Delta L = 1-|A_R^{un}|^2=\tau_{soj}/\tau. \qquad (36)$$

So, taking into account (34), we obtain:

$$\tau_{soj}/\tau=(k_1/k_2)\,|A_T^{ex}|\ \text{and}\ \tau_{soj}=(k_1/k_2)\,|A_T^{in}|^2\tau=P\tau, \qquad (37)$$

where $P=(k_1/k_2)|A_T|^2$.

Formula (37), rewritten, with $\nu=1/\tau_{so}$ and $\tau=1/\lambda$, in the form of the known formula (see, for instance, [3, 4])

$$\lambda=\nu P \qquad (38)$$

represents a new phenomenological approach to the meaning of the pre-exponential factor.

Let us calculate τ_{soj}, for instance, for ^{210}Po (for values E_0 = 5.407 MeV, V_0 = 16.7 MeV, R_1 = = 8.76 fm and R_1 = 8.975 fm), taking into account that τ = 138.376 days = 11955686.4 s. Then we obtain $\tau_{soj}=2.434\cdot10^{-20}$ s and $\tau_{soj}=5.740\cdot10^{-20}$ s, and consequently now $\nu=1/\tau_{soj}=4.108\cdot10^{19}\ \text{s}^{-1}$ and $1.742\cdot10^{19}\ \text{s}^{-1}$, respectively.

Obtained here quantity $\nu=1/\tau_{soj}$ is different from the Gamow pre-exponential factor $v_0/2R_1\equiv[2(E+V_0)/m]^{1/2}/2R_1$, evaluated as a simple classical number of purely kinetic impacts on the nuclear surface per time unit, and also from the Landau evaluation of the pre-exponential factor as $D/2\pi\hbar$ (D is the mean level distance for the parent nucleus in the considered energy interval).

Numerically our value is strongly different from the Gamow value: for chosen in [16] E = = 5.407 MeV, V_0 = 16.7 MeV, R_1 = 8.76 fm and R_1 = 8.975 fm, $v_0/2R_1=1.881\cdot10^{21}\ \text{s}^{-1}$ and $1.836\cdot10^{21}\ \text{s}^{-1}$, respectively. Such difference can be explained physically by the very small moving time inside the potential well in comparison with the duration of the vanishing and formation processes. Also it is easy to see that the value of R_0 does not essentially influence on the values of A_R, A_T and τ_{soj}.

For D = 100 keV, the value $D/2\pi\hbar = 2.418\cdot10^{19}$ s^{-1}. So, our results are not very distant from the Landau value (may be, because both quantities are connected with the fundamental intrinsic properties of the internal motions inside the parent nucleus).

By the way, for the rectangular barrier (see Fig. 3, *a*) the scattering phase time is

$$\tau_{sc}^{Ph} = \hbar \frac{\partial \arg Se^{ik_1R_2}}{\partial E} + \frac{R_2}{v_1} \quad (39)$$

and the limit $\chi(R_2 - R_1)$ approaching infinity it goes to $2/(v\chi)$.

So in this limit, the scattering phase time coincides with $\tau_R^{Ph(ex)}$ (see, for instance, [1, 2]).

Conclusions

The obtained here results (together with the previous results from [7]) can be used as an initial phase for the time-dependent study of nuclear reactions (beginning from the one-channel elastic scattering) and decays for any value of l and also for non-spherical cases. Also they can be used as an initial phase for analysis of the sub-barrier low-energy astrophysical nuclear-fusion reactions. In the last case it is important to take into account not only the penetration factor (10) but also the factor $\frac{1}{1+A_R^{in}}$ which appear in the consequence of the multiple internal reflections between the hard core and the diameter-opposite points of the internal spherical 3D barrier wall. These both factors are absent in 1D approximation, although in many papers (see, for instance, [8 - 12] and appropriate references therein) this 1D approximation is typical.

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ТУНЕЛЮВАННЯ ТА РОЗСІЯННЯ ЧАСТИНОК У ТРИВИМІРНОМУ ПОТЕНЦІАЛІ З ЖОРСТКОЮ СЕРЦЕВИНОЮ ТА ЗОВНІШНІМ ПОТЕНЦІАЛЬНИМ БАР'ЄРОМ

В. С. Ольховський, М. В. Романюк

Вивчено тунелювання та розсіяння нерелятивістської частинки крізь сферичний тривимірний потенціальний бар'єр (прямокутний чи кулонівський), що містить потенціальну прямокутну яму й жорстку серцевину всередині. Уперше було отримано явні аналітичні вирази для S-матриці пружного розсіяння та амплітуд імовірності (зовнішнього та внутрішнього відбиття, тунелювання ззовні та тунелювання із середини) і співвідношення між ними. У роботі, на відміну від типового спрощеного одновимірного наближення, яким користуються в низькоенергетичних астрофізичних реакціях синтезу, ми підкреслюємо необхідність розгляду тривимірної картини, яка дає багаторазові внутрішні відбивання від внутрішньої стінки бар'єра, а також більш точний коефіцієнт проникнення.

Ключові слова: тривимірне тунелювання, розсіяння, бар'єр, потенціальна яма, жорстка серцевина.

ТУННЕЛИРОВАНИЕ И РАССЕЯНИЕ ЧАСТИЦ В ТРЕХМЕРНОМ ПОТЕНЦИАЛЕ С ЖЕСТКОЙ СЕРДЦЕВИНОЙ И ВНЕШНИМ ПОТЕНЦИАЛЬНЫМ БАРЬЕРОМ

В. С. Ольховський, М. В. Романюк

Изучено туннелирование и рассеяние нерелятивистской частицы сквозь сферический трехмерный потенциальный барьер (прямоугольный или кулоновский), содержащий потенциальную прямоугольную яму и жесткую сердцевину внутри. Впервые были получены явные аналитические выражения для S-матрицы упругого рассеяния и амплитуд вероятности (внешнего и внутреннего отражения, туннелирования внутрь и туннелирования из середины) и соотношения между ними. В работе, в отличие от типичного упрощенного одномерного приближения, каким пользуются в низкоэнергетических астрофизических реакциях синтеза, мы подчеркиваем необходимость рассмотрения трехмерной картины, которая описывает многократные внутренние отражения от внутренней стенки барьера, а также более точный коэффициент проникновения.

Ключевые слова: трехмерное туннелирование, рассеяние, барьер, потенциальная яма, жесткая сердцевина.

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