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EXTENDED SYMMETRIES OF THE KINETIC PLASMA THEORY MODELS

V. B. Taranov

Institute for Nuclear Research, National Academy of Sciences of Ukraine, Kyiv

Symmetry extension of the kinetic theory of collisionless plasma containing particles with equal charge to mass ratio is considered. It is shown that this symmetry allows us to reduce the number of equations. Symmetries obtained for the integro-differential equations of the kinetic theory by the indirect algorithm are compared to those obtained by direct methods. The importance of additional conditions – positiveness and integrability of distribution functions, existence of their moments - is underlined.

1. Introduction

In recent decades the Lie group analysis has been applied to explore many physically interesting nonlinear problems in gas dynamics, plasma physics etc. [1]. Furthermore, different extensions of the classical Lie algorithm to the integro-differential systems of equations of kinetic theory were proposed [2 - 4]. In particular, this can be done by using three step algorithm [2, 3] which allows us to obtain symmetries of the kinetic equations from the symmetries of an infinite set of partial differential equations for the moments of distribution functions. First, we determine the symmetry of the truncated system of equations for the moments by the usual Lie procedure. Then, we extend the obtained symmetry transforms to the case of an infinite set of the equations. Finally, we restore the symmetries of the integro-differential kinetic theory equations using the generating functional of the above mentioned moments.

By this indirect algorithm, it was shown in [2] that the integro-differential equations of the kinetic theory of collisionless plasma containing particles with equal charge to mass ratio have additional symmetries.

In the Section 2, kinetic model for the collisionless plasma is presented. In the Section 3, extended symmetry transformations obtained by the indirect algorithm are discussed for the plasma containing alpha particles and deuterium ions. It is shown that these transformations allow us to reduce the number of equations. Then, further symmetry extension is considered. The role of additional conditions – positiveness and integrability of the distribution functions, existence of the moments of these functions – is discussed. Conclusions are made in the Section 4.

2. Collisionless plasma model

Let us consider N-component collisionless plasma. In this case Vlasov - Maxwell integro-differential system of equations holds:

$$\begin{gathered}\partial f_\alpha/\partial t + \mathbf{v} \cdot \partial f_\alpha/\partial \mathbf{r} + (e_\alpha/m_\alpha)[\mathbf{E} + (1/c)(\mathbf{v} \times \mathbf{B})] \cdot \partial f_\alpha/\partial \mathbf{v} = 0,\\ \nabla \times \mathbf{E} + (1/c)\, \partial \mathbf{B}/\partial t = 0, \quad \nabla \cdot \mathbf{E} = 4\pi\, \rho,\\ \nabla \times \mathbf{B} = (1/c)\, \partial \mathbf{E}/\partial t + (4\pi/c)\, \mathbf{j}, \quad \nabla \cdot \mathbf{B} = 0.\end{gathered} \tag{1}$$

where $f_\alpha(t, \mathbf{r}, \mathbf{v})$ is the distribution function of the α-th component of the plasma. Charge and mass of particles of the α-th component are denoted by e_α and m_α , respectively.

Charge and current densities have the form

$$\begin{gathered}\rho = \int (\Sigma e_\alpha f_\alpha)\, d\mathbf{v} \equiv \Sigma\, e_\alpha\, M_{\alpha,\,0},\\ \mathbf{j} = \int \mathbf{v}\, (\Sigma e_\alpha f_\alpha)\, d\mathbf{v} \equiv \Sigma\, e_\alpha\, \mathbf{M}_{\alpha,\,1},\end{gathered} \tag{2}$$

where Σ means the sum over α from 1 to N.

Moments $M_{\alpha,0}$ and $\mathbf{M}_{\alpha,1}$ of the distribution functions f_α are very important since they are present in Maxwell equations explicitly. Higher moments are determined in a similar way:

$$M_{\alpha,\mathbf{k}}(t, \mathbf{r}) \equiv \int (v_x)^{k1} (v_y)^{k2} (v_z)^{k3} f_\alpha(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}, \tag{3}$$

where $\mathbf{k} \equiv \{k1, k2, k3\}$. Distribution functions f_α must be non-negative and integrable. At least the moments $M_{\alpha,0}$ and $\mathbf{M}_{\alpha,1}$ must exist.

3. Extended symmetries of the model

It was shown in [2] that if there are plasma components with equal charge to mass ratio of particles,

$$(e_\mu/m_\mu) = (e_\nu/m_\nu)$$

for some μ and ν, kinetical model (1), (2) admits additional symmetries with infinitesimal operators

$$X_{\mu\nu} = f_\mu\, (\partial/\partial f_\mu - (e_\mu/e_\nu)\, \partial/\partial f_\nu). \tag{4}$$

It is known that alpha particles and deuterium ions participating in a thermonuclear reaction

$$D + T \rightarrow He + n + 17{,}6 \text{ MeV}$$

have close charge to mass ratios which can be treated as equal in the classical physics context.

Let $\alpha = 1$ and $\alpha = 2$ correspond to alpha particles and deuterium ion components, respectively, so we assume (and it is a good approximation)

$$e_1/m_1 = e_2/m_2. \tag{5}$$

Then the equations (1), (2) admit the following additional symmetries:

$$X_{12} = f_1\, \partial/\partial f_1 - (e_1 / e_2)\, f_1\, \partial/\partial f_2, \quad X_{21} = f_2\, \partial/\partial f_2 - (e_2 / e_1)\, f_2\, \partial/\partial f_1. \tag{6}$$

Finite transformations generated by X_{12}, X_{21} are

$$f_1' = f_1 \exp(a_{12}), \quad f_2' = f_2 + (e_1/e_2)\,(1 - \exp(a_{12}))\, f_1 \tag{7}$$

and

$$f_2' = f_2 \exp(a_{21}), \quad f_1' = f_1 + (e_2/e_1)\,(1 - \exp(a_{21}))\, f_2 \tag{8}$$

respectively, a_{12} and a_{21} are arbitrary constants.

It is clear that if the moments $M_{\alpha,\mathbf{k}}(t, \mathbf{r})$ exist for the functions f_1, f_2, then the same holds for the transformed functions f_1', f_2'. On the other hand, for positive definite f_1, f_2 transformed functions f_1', f_2' can be negative for some values of the parameters a_{12} and a_{21}.

It is important that in the limit $a_{12} \rightarrow -\infty$ we obtain from (7) the transformation

$$f_1' = 0, \quad f_2' = f_2 + (e_1/e_2)\, f_1. \tag{9}$$

According to (9), we can solve the system (1), (2), omitting its first equation

$$\partial f_1/\partial t + \mathbf{v}\, \partial f_1/\partial \mathbf{r} + (e_1/m_1)\, (\mathbf{E} + (1/c)\, (\mathbf{v} \times \mathbf{B}))\, \partial f_1/\partial \mathbf{v} = 0, \tag{10}$$

and considering f_2' (which is normalized charge density of the first two plasma components, $((e_1 f_1 + e_2 f_2)/ e_2)$ as the new distribution function for the second plasma component. The first plasma component remains present in the solution process only in the initial condition for f_2'. When the reduced system of equation will be solved, we must return to the equation (10), where $\mathbf{E}$ and $\mathbf{B}$ are now known functions, and solve this linear equation for f_1.

The system (1), (2) is even more symmetric. It can be readily shown that the following transformation:

$$f_1' = f_1 - e_2 F(f_1, f_2), \quad f_2' = f_2 + e_1 F(f_1, f_2), \tag{11}$$

where $F(f_1, f_2)$ is an arbitrary function of its arguments, leave the Vlasov - Maxwell equations invariant, if we perform in (2) the summation first and the integration later.

The transformation (11) does not preserve, in general, the positiveness of the distribution functions. Moreover, it can lead to the divergence of the moments, which can be shown, for example, by choosing F = const.

Another difficulty in treating (11) as a symmetry originate from the fact that no explicit transform can be obtained, in general, from (11) for the moments $M_{\alpha,k}(t, \mathbf{r})$.

4. Conclusions

Symmetry extension for the plasma containing the components with closed values of the charge to mass ratio of particles was considered. As an example the plasma containing alpha particles and deuterium ions was choosen.

If we explore the symmetry of the integro-differential kinetic equations (1), (2) by some of direct methods (1), ignoring additional conditions like that: distribution functions must be non-negative; they must be integrable; their moments $M_{\alpha,0}$, $\mathbf{M}_{\alpha,1}$ and some higher moments must exist we obtain the largest symmetry extension, namely the transform (11) determined by the arbitrary function $F(f_1, f_2)$

$$f_1' = f_1 - e_2 F(f_1, f_2), \quad f_2' = f_2 + e_1 F(f_1, f_2).$$

In the simplest case of this transformation, F = const, moments $M_{\alpha,0}$ diverge.

If we explore the symmetry of (1), (2) by the indirect algorithm [2, 3] based on the use of the infinite set of moment equations, all transformations producing divergent moments are automatically excluded. We obtain symmetries (7), (8), which are the linear homogeneous transformations of f_1, f_2, i. e.

$$f_1' = a_{11}f_1 + a_{12} f_2 , \quad f_2' = a_{21}f_1 + a_{22} f_2$$

where a_{ij} are constants, preserving the invariant form – charge density of the considered plasma components

$$e_1 f_1 + e_2 f_2. \tag{12}$$

Symmetries (7), (8) produce simple linear transformations of all the moments, they cannot lead to the divergence of the moments. So the conditions (b) and (c) are satisfied automatically. On the other hand, these symmetries can violate the condition (a) of perturbation functions non negativeness for some values of the constants a_{12} and a_{21}. As a consequence, group orbits can contain simultaneously physical (non-negative) and non-physical solutions.

Symmetries (7), (8) are particular examples of the general symmetry (11) determined by the special choice of the function F, namely

$$F = (1 - \exp(a_{12}))\, f_1/e_2 \text{ for (7) and } F = -(1 - \exp(a_{21}))\, f_2/e_1 \text{ for (8).}$$

Finally, if we choose the invariant (charge density) (12) as a new distributiom function in (1), (2), the number of equations in this system will be reduced by 1. This procedure does not violate any of physical conditions (a) – (c). Of course, new distribution function can be normalized like f_2' in (9).

In general, the presented results made more clear, what can we expect from the symmetry considerations of the kinetic plasma theory models by different, direct or indirect, methods.

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РОЗШИРЕНІ СИМЕТРІЇ МОДЕЛЕЙ КІНЕТИЧНОЇ ТЕОРІЇ ПЛАЗМИ

В. Б. Таранов

Розглянуто розширення симетрії кінетичної теорії плазми без зіткнень, що складається з частинок з однаковим відношенням заряду до маси. Показано, що така симетрія дозволяє зменшити кількість рівнянь. Симетрії, одержані для інтегро-диференціальних рівнянь кінетичної теорії непрямим алгоритмом, порівнюються з такими, що отримані прямими методами. Підкреслено важливість додаткових умов – позитивності та інтегрованості функцій розподілу, існування їхніх моментів.

РАСШИРЕННЫЕ СИММЕТРИИ МОДЕЛЕЙ КИНЕТИЧЕСКОЙ ТЕОРИИ ПЛАЗМЫ

В. Б. Таранов

Рассмотрено расширение симметрии кинетической теории плазмы без столкновений, содержащей частицы с одинаковым отношением заряда к массе. Показано, что такая симметрия позволяет уменьшить количество уравнений. Симметрии, полученные для интегро-дифференциальных уравнений кинетической теории непрямым алгоритмом, сравниваются с такими, которые получены прямыми методами. Подчеркнута важность дополнительных условий – позитивности и интегрированости функций распределения, существования их моментов.

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