УДК 533.9

SELF SIMILAR ASYMPTOTICS OF THE DRIFT ION ACOUSTIC WAVES

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A 3D model for the coupled drift and ion acoustic waves is considered. It is shown that self-similar solutions can exist due to the symmetry extension in asymptotic regimes. The form of these solutions is determined in the presence of the magnetic shear as well as in the shearless case. Some of the most symmetric exact solutions are obtained explicitly. In particular, solutions describing asymptotics of zonal flow interaction with monochromatic waves are presented and corresponding frequency shifts are determined.

Introduction

Low frequency drift and ion acoustic waves play an important role in the transport processes in magnetized plasmas [1]. The main problem in their treatment is the presence of nonlinear effects even at relative small amplitudes. Up to now, many results were obtained by traditional methods - non-symmetric perturbation theory, wave-kinetic equations technique supported by numerical simulations (see the review article [1] and citations therein), but many questions remain open. Namely, only in the particular case of pure ion-acoustic waves dispersion and nonlinear terms can balance to form Korteweg-de Vries solitons. In the two-dimensional case of pure drift waves (Hasegawa-Mima model) anisotropic dispersion fails to balance degenerated vortex nonlinearity terms, so Larichev-Reznik type vortex structures (modons) does not have soliton stability properties. Furthermore, dynamical chaos and fractional kinetics are also important in the evolution of the drift ion acoustic waves [2, 3].

In this situation, symmetry considerations can help us to better understand the properties of the coupled drift ion acoustic waves. We can expect to find some exact invariant solutions or to determine the form of the functions describing these solutions. We can also build symmetric perturbation theory (see, e.g. [4 - 7]).

In the previous work [8, 9], a spatially three-dimensional model (see, e.g. [10]) was considered for the coupled drift and ion acoustic waves. Symmetry analysis for this model was performed. The influence of the magnetic shear on the symmetry properties was studied. The form was determined of the most symmetric localized and spatially periodic waves. For the waves of small but finite amplitude perturbation theory based on the multiple-time-scale formalism was built. Some precise and perturbative solutions describing higher harmonics generation, frequency shifts and zonal flow generation by initially monochromatic waves were obtained.

In the present work the asymptotic solutions of this model are considered. The model is shortly described in the first section. In the second section symmetries and self-similar asymptotic solutions are presented. In the last section conclusions are drawn.

Model

Let us consider an inhomogeneous plasma slab with the background plasma density $n_o \sim \exp(x / L_n)$ in the external magnetic field $\mathbf{B} = B_o (\mathbf{e}_z + \mathbf{e}_y (x / L_{sh}))$ with the shear length L_{sh}. Electrons, unlike ions, are magnetized, smoothing an electrostatic potential Φ along the magnetic field lines. In this case, 3D generalization [10] of the Hasegawa-Mima model equations holds:

$$\partial\Psi/\partial t + J[\Phi, \Psi] + \partial v/\partial z + S x\, \partial v/\partial y = \partial\Phi/\partial y,$$

$$\partial v/\partial t + J[\Phi, v] + \partial\Phi/\partial z + S x\, \partial\Phi/\partial y = 0, \quad (1)$$

$$\Psi = \Phi - \Delta_{tr}\Phi,$$

where v is the ion velocity along the main magnetic field direction $\mathbf{e_z}$, $\Psi \equiv \Psi_z$ is the only non-zero component of the generalized vorticity, $S = L_{sh} / L_n$. the operators in (1) are equal to

$$J[F, G] \equiv \partial F/\partial x\ \partial G/\partial y - \partial F/\partial y\ \partial G/\partial x, \quad \Delta_{tr}\ \Phi \equiv \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2.$$

The system (1) is written in dimensionless variables

$$\varepsilon\omega_{ci}t,\ x/r_B,\ y/r_B,\ \varepsilon z/r_B,\ e\Phi/T_e\varepsilon,\ v_{iz}/c_s,$$

where the ion cyclotron frequency $\omega_{ci} = eB_o/Mc$ and ion sound speed $c_s = (T_e/M)^{1/2}$ determine the characteristic dispersion length $r_B = c_s / \omega_{ci}$ and the small parameter ε is equal to the ratio r_B / L_n.

Hydrodynamic ion component velocity (transverse to the main external magnetic field) is determined by the expressions

$$v_{ix} = - \partial\Phi/\partial y, \quad v_{iy} = \partial\Phi/\partial x.$$

The transverse electric field components are

$$E_x = - \partial\Phi/\partial x, \quad E_y = - \partial\Phi/\partial y.$$

Pure ion acoustic (potential) nonlinearity is not present in the model (1) as well as other higher order effect in the parameter ε.

Symmetries and self-similar solutions

The presence of the parameter r_B in the model (1) leads to the symmetry reduction and, as a consequence, to the absence of self-similar solutions. Nevertheless, self-similar solutions can exist as asymptotic ones. To prove this, let us first neglect the magnetic shear terms (i.e. let us put S = 0) and the term $\Delta_{tr}\ \Phi$ in the system (1). In this way we obtain more symmetric equations, which admit the following similarity transforms:

$$X_1 = x\ \partial/\partial x + v\ \partial/\partial v + \Phi\ \partial/\partial\Phi + \Psi\ \partial/\partial\Psi, \tag{2}$$

$$X_2 = y\ \partial/\partial y + z\ \partial/\partial z + t\ \partial/\partial t. \tag{3}$$

It can be readily seen that only the second transformation, X_2 remains in the presence of magnetic shear ($S \neq 0$). Moreover, in this case non-trivial self-similar solutions can exist only if Laplacian operator is reduced to $\Delta_{tr}\ \Phi \equiv \partial^2\Phi/\partial x^2$. In other words, an additional condition

$$\partial^2\Phi/\partial y^2 = 0 \tag{4}$$

must be imposed on the solutions of (1). Then, symmetry X_2 determines the form of the corresponding self-similar solutions:

$$\Phi = F(x, z/t)\ (y/t) + G(x, z/t), \quad v = U(x, y/t, z/t). \tag{5}$$

The RHS functions F(x, z/t), G(x, z/t) and U(x, y/t, z/t) must be found from the system (1). The form of the vorticity function Ψ is determined by the last equation of the system (1):

$$\Psi = (F - \partial^2F/\partial x^2)\ (y/t) + (G - \partial^2G/\partial x^2).$$

In the shearless case, S = 0 the self-similar solutions of the form (5) still exist and are determined by the system (1) with S = 0.

Other self-similar solutions are possible in the absence of magnetic shear, S = 0, since the additional symmetry X_1 allows us to try solutions in the form:

$$\Phi = x\ F\ (t, y, z), \quad \Psi = x\ G\ (t, y, z), \quad v = x\ U(t, y, z). \tag{6}$$

Based on these solutions the Laplacian operator is automatically reduced to $\Delta_{tr}\,\Phi \equiv \partial^2\Phi/\partial y^2$. Inserting Φ, Ψ and v in the form determined by (6) in the system (1) and setting S = 0, we readily obtain:

$$\partial G/\partial t + F\,\partial G/\partial y - (G+1)\,\partial F/\partial y + \partial U/\partial z = 0,$$
$$\partial U/\partial t + F\,\partial U/\partial y - U\,\partial F/\partial y + \partial F\partial z = 0, \quad (7)$$
$$G = F - \partial^2 F/\partial y^2.$$

For the pure ion acoustic waves ($\partial/\partial y = 0$), the system (7) is reduced to the linear ion acoustic wave equations for F and its conjugate U:

$$\partial F/\partial t + \partial U/\partial z = 0, \quad \partial U/\partial t + \partial F\partial z = 0, \quad G = F. \quad (8)$$

For the pure drift waves (Hasegawa-Mima model, $\partial\partial z = 0$, U = 0) the system (7) becomes

$$\partial G/\partial t + F\,\partial G/\partial y - (G+1)\,\partial F/\partial y = 0, \quad (9)$$
$$G = F - \partial^2 F/\partial y^2$$

and can be transformed to the form

$$\partial(1/(G+1))/\partial t + \partial(F/(G+1))/\partial y = 0, \quad G = F - \partial^2 F/\partial y^2. \quad (10)$$

The system (10) is still nonlinear, but we can use its symmetries to find some exact solutions. Namely, due to continuous symmetries

$$X_3 = \partial/\partial t, \quad X_4 = \partial/\partial y \quad (11)$$

and to the discrete ones, $t \to -t$, $y \to -y$, we can find exact invariant solutions describing:

a) homogeneous zonal flow, F = G = const;

b) monochromatic wave: $F = A\cos(\omega t + q y)$, amplitude A is an arbitrary constant and the frequency is determined by the dispersion relation of the linear theory

$$\omega = q/(1+q^2); \quad (12)$$

c) interaction of the zonal flow with the monochromatic wave

$$F = \alpha\,(1 + \beta\cos((\omega + \delta\omega)\,t + q\,y)), \quad (13)$$

where α is arbitrary constant amplitude, constant factor β determines the relative weight of the zonal flow and the monochromatic wave, the frequency is $\omega = q/(1+q^2)$ and the frequency shift is

$$\delta\omega = -\,\alpha\,q^3/(1+q^2). \quad (14)$$

If the amplitude α is equal to $1/q^2$, the frequency is shifted to zero and we obtain time independent solution

$$F = (1 + \beta\cos(q\,y))/q^2. \quad (15)$$

In the general case of the coupled drift and ion acoustic waves, $\partial\partial z \neq 0$, $U \neq 0$, self-similar solutions of the form (6) can be obtained as perturbation theory solutions of (7). They can be readily found in the limit $k_1 x \to 0$ from the more general solutions obtained in the previous work [8, 9].

Conclusions

It is shown that in asymptotic regimes the symmetry of the system (1) is extended, namely the scaling transformations (2) and (3) appear. The physical reason for this symmetry extension is that nonlinear waves have a simpler shape near their critical points – nodes, crests etc. Due to the symmetry extension, self-similar solutions of the form (5) and (6) become possible.

Self-similar solutions (5) describe the nonlinear drift ion acoustic wave in the vicinity of the maximum electric field component E_y in the drift direction. These solutions are possible in the presence of the magnetic shear as well as in the shearless case.

Self-similar solutions (6) describe the non-linear drift ion acoustic wave in the vicinity of its nodal points. Such solutions are possible only in the shearless case, S = 0. In general, these solutions can be obtained by perturbative treatment of the system (7). First two orders of amplitude self-similar solutions coincide with the $k_1x \to 0$ limit of the previous more general results [8, 9]. In the particular case of pure drift waves self-similar analytical solution describing the interaction of the zonal flow with the monochromatic wave is obtained (see (12) – (14))

$$\Phi = \alpha\, x\, (1 + \beta \cos((\omega + \delta\omega)\, t + q\, y)), \qquad (16)$$

where the frequency ω is determined by the dispersion relation of the linear theory (12) and the frequency shift to the frequency ratio is

$$(\delta\omega / \omega) = -\, \alpha\, q^2. \qquad (17)$$

When the amplitude α reaches the value $1/q^2$, wave frequency is shifted to zero.

The solutions obtained in this work can be useful for plasma diagnostics and for testing numerical codes modeling the drift ion acoustic waves. Investigations of their stability can be helpful in the study of the dynamical chaos which is present in the model considered above.

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АВТОМОДЕЛЬНІ АСИМПТОТИКИ ДРЕЙФОВО-ІОННОАКУСТИЧНИХ ХВИЛЬ

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Розглядається тривимірна модель для пов'язаних між собою дрейфових та іонноакустичних хвиль. Показано, що завдяки поширенню симетрії в асимптотичних режимах можуть існувати автомодельні рішення. Визначено форму таких рішень як у присутності магнітного широ, так і в безшировому випадку. Деякі з найбільш симетричних точних рішень одержано в явному вигляді. Зокрема, одержано рішення, що представляють взаємодію зональних потоків із монохроматичними хвилями, і визначено відповідні зсуви частоти.

АВТОМОДЕЛЬНЫЕ АСИМПТОТИКИ ДРЕЙФОВО-ИОННОАКУСТИЧЕСКИХ ВОЛН

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Рассмотрена трехмерная модель для связанных дрейфовых и ионноакустических волн. Показано, что благодаря расширению симметрии в асимптотических режимах могут существовать автомодельные решения. Определена форма таких решений как в присутствии магнитного шира, так и в бесшировом случае. Некоторые из наиболее симметричных точных решений найдены в явном виде. В частности, получены решения, описывающие взаимодействие зональных потоков с монохроматическими волнами, и определены соответствующие сдвиги частоты.

Received 29.04.04,
revised - 07.07.04.