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DRIFT ION ACOUSTIC WAVES AND THEIR SYMMETRIES

V. B. Taranov

Institute for Nuclear Research, National Academy of Sciences of Ukraine, Kyiv

3D model for the coupled drift and ion acoustic waves is considered. Symmetries of the model are found in the presence of the magnetic shear as well as in the shearless case. Some of the most symmetric solutions, exact and perturbative, are presented. In particular, solutions describing zonal flow generation by initially monochromatic waves are obtained.

Introduction

Low frequency drift and ion acoustic waves play an important role in the transport processes in magnetized plasmas [1]. The main problem in their treatment is the presence of nonlinear effects even at relative small amplitudes. In this situation the symmetry analysis can help us to find exact or perturbative solutions (see, e.g. [2]).

Only in the particular case of pure ion acoustic waves (the well-known Korteweg-de Vries equation in one dimension) dispersion and nonlinear steepening can balance to form coherent structures called solitons. In the two-dimensional case of pure drift waves (Hasegawa-Mima model) anisotropic dispersion fails to balance degenerated vortex nonlinearity terms [2, 3].

In the present work more general spatially three-dimensional model [4] is considered for the coupled drift and ion acoustic waves. Symmetry analysis for this model is performed. Magnetic shear influence on the symmetry properties is studied. The form is determined of the most symmetric localized and spatially periodic waves. For the waves of small but finite amplitude the perturbation theory based on multiple-time-scale formalism is built. Some exact and perturbative solutions describing higher harmonics generation, frequency shifts and zonal flow generation by initially monochromatic waves are presented.

Model

Let us consider an inhomogeneous plasma slab with the background plasma density $n_o \sim \exp(x / L_n)$ in the external magnetic field $\mathbf{B} = B_o (\mathbf{e}_z + \mathbf{e}_y (x / L_{sh}))$ with the shear length L_{sh}. Electrons, unlike ions, are magnetized, smoothing an electrostatic potential Φ along the magnetic field lines. In this case, 3D generalization [4] of the Hasegawa-Mima model equations holds:

$$d\Psi/dt + dv/dz = \partial\Phi/\partial y,$$

$$dv/dt + d\Phi/dz = 0, \qquad (1)$$

$$\Psi = \Phi - \Delta_{tr}\Phi,$$

where v is the ion velocity along the magnetic field direction Oz, $\Psi \equiv \Psi_z$ is the only non-zero component of the generalized vorticity. The operators in (1) are

$$d/dt = \partial/\partial t + J[\,\Phi, \dots\,], \quad J[\,F, G\,] \equiv \partial F/\partial x\, \partial G/\partial y - \partial F/\partial y\, \partial G/\partial x,$$

$$d/dz = \partial/\partial z + S\, x\, \partial/\partial y, \quad S = L_{sh} / L_n, \quad \Delta_{tr}\Phi \equiv \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2.$$

The system (1) is written in dimensionless variables $\varepsilon\omega_B t$, x/r_B, y/r_B, $\varepsilon z/r_B$, $e\Phi/T_e\varepsilon$, where the ion cyclotron frequency $\omega_{ci} = eB_o/Mc$ and ion sound speed $c_s = (T_e/M)^{1/2}$ determine the characteristic dispersion length $r_B = c_s / \omega_{ci}$ and the small parameter ε is equal to the ratio r_B / L_n.

Symmetries

First, to express the symmetries in more simple form, let us perform the transformation

$$\Phi = \Phi(t, x, y + t, z) - x, \qquad v = v(t, x, y + t, z)$$

which removes the term $\partial\Phi/\partial y$ from the first equation of the system (1).

In the shearless case, S = 0, we obtain Lie group of symmetry by the standard procedure. This group is as follows:

$$X_1 = \partial/\partial t\,,\quad X_2 = \partial/\partial x\,,\quad X_3 = \partial/\partial y\,,\quad X_4 = \partial/\partial z,$$

$$X_5 = - y\, \partial/\partial x + x\, \partial/\partial y,$$

$$X_6 = t\, \partial/\partial t + z\, \partial/\partial z - \Phi\, \partial/\partial\Phi - \Psi\, \partial/\partial\Psi - v\, \partial/\partial v, \tag{2}$$

$$X = F(t, z)\, (\partial/\partial\Phi + \partial/\partial\Psi) + G(t, z)\, \partial/\partial v,$$

where F and G satisfy the linear wave equation in variables t and z:

$$\partial F/\partial t + \partial G/\partial z = 0 \quad \text{and} \quad \partial G/\partial t + \partial F/\partial z = 0$$

The model (1) also admits the reflection symmetries

$$\begin{aligned} &a) \quad \{ x, \Phi, \Psi, v \} \to \{ - x, - \Phi, - \Psi, - v \} \\ &b) \quad \{ t, y, v \} \to \{ - t, - y, - v \} \\ &c) \quad \{ z, v \} \to \{ - z, - v \} \end{aligned} \tag{3}$$

In the presence of magnetic shear, $S \neq 0$, the symmetry group is reduced to

$$X_1\,,\quad X_3\,,\quad X_4\,,\quad \text{and}\quad X \tag{4}$$

and to the less number of the reflection symmetries

$$\begin{aligned} &a) \quad \{ x, z, \Phi, \Psi \} \to \{ - x, -z, - \Phi, - \Psi \} \\ &b) \quad \{ t, y, z \} \to \{ - t, - y, - z \}. \end{aligned} \tag{5}$$

The physical reason for this symmetry reduction is the explicit and anisotropic dependence of the external magnetic field on the x-coordinate, i.e. along the background plasma density gradient.

Solutions

Let us review now the most symmetric solutions of the model (1) in the shearless case (S = 0).

a) Pure drift waves, $\partial/\partial z = 0$, periodic in variables x and y solutions. In this particular case we have two exact solutions, namely the periodic zonal flow of the plasma, $\Phi = \sin(k_1 x)$ and the monochromatic standing wave:

$$\Phi = \sin(k_1 x) \cos(\omega_1 t + k_2 y).$$

Amplitudes of these solutions are arbitrary, since the nonlinear term exactly vanishes.

On the combination of these exact solutions the nonlinear term is not zero, so we obtain the perturbative solution describing drift wave interaction with a zonal flow:

$$\Phi = \alpha\, \Phi_1 + \alpha^2\, \Phi_2 + \ldots \tag{6}$$

$$\Phi_1 = (1 + \beta \cos(\omega_1 t + k_2 y + \delta\omega\, t) \sin(k_1 x),$$

where $\omega_1 = k_2 /(1 + k_1^2 + k_2^2)$ and $\delta\omega = (\alpha^2/12)\, k_2^3\, (3 k_1^2 + k_2^2)$. The constant β is the weight parameter of the standing wave relative to the zonal flow.

$$\Phi_2 = \beta\, k_2^2\, (1 + k_1^2 + k_2^2)\, (\cos(\omega_2 t + k_2 y) - \cos(\omega_1 t + k_2 y))\, \sin(2k_1 x) / (6k_1),$$

where $\omega_2 = k_2 /(1 + 4k_1^2 + k_2^2)$.

As a result of the interaction, higher harmonics are generated, beginning from the second order α^2 of amplitude. Frequency shift $\delta\omega$ of the main harmonic appears as the third order effect ($\sim \alpha^3$). Pulsations of the zonal flow also appear in the third order, the corresponding complicated expressions are omitted here.

b) 3D nonlinear drift ion acoustic standing wave, which potential Φ is odd in the plasma inhomogeneity direction x and even along the external magnetic field (variable z) and on the combination of time and drift variables t, y. Additional nonlinear term $J[\,\Phi, v\,]$ is not zero and we must to built the perturbative solution:

$$\Phi = \alpha\, \Phi_1 + \alpha^2\, \Phi_2 + \ldots, \quad v = \alpha\, v_1 + \alpha^2\, v_2 + \ldots \qquad (7)$$

$$\Phi_1 = \sin(k_1 x)\, (\omega_1 \cos(\theta) \cos(\omega_1 t + k_2 y) + \omega_2 \sin(\theta) \cos(\omega_2 t + k_2 y))\cos(k_3 z)$$

$$v_1 = k_3 \sin(k_1 x)\, (\cos(\theta) \sin(\omega_1 t + k_2 y) + \sin(\theta) \sin(\omega_2 t + k_2 y)) \sin(k_3 z),$$

where the parameter θ determines the relative weight of the components with the frequencies (in linear approximation) ω_1 and ω_2:

$$\omega_{1,2} = (k_2 \pm (k_2^2 + 4(1 + k_1^2 + k_2^2)\, k_3^2)^{1/2}) / (2\, (1 + k_1^2 + k_2^2)).$$

In the second order of amplitude higher harmonics are generated

$$\Phi_2 = (k_1 k_2/16) \sin(2k_1 x) \cos(2k_3 z)\, (2(\omega_1 + \omega_2 + (\omega_1 - \omega_2) \cos(2\theta) \sin^2(\omega_3 t/2) -$$

$$- ((\omega_1 + \omega_2)\, \omega_3^2\, (\cos((\omega_1 - \omega_2)\, t) - \cos(\omega_3 t)) \sin(2\theta))/((\omega_1 - \omega_2)^2 - \omega_3^2)),$$

$$v_2 = - (k_1 k_2/16)(1 + 4k_1^2)^{1/2} \sin(2k_1 x) \sin(2k_3 z)\, ((\omega_1 + \omega_2 + (\omega_1 - \omega_2) \cos(2\theta)) \sin(\omega_3 t) +$$

$$+ (\omega_1 + \omega_2)\, \omega_3 \sin(2\theta)\, ((\omega_1 - \omega_2)\, (\sin((\omega_1 - \omega_2)\, t) - \omega_3 \sin(\omega_3 t))/((\omega_1 - \omega_2)^2 - \omega_3^2)),$$

where $\omega_3 = 2k_3/(1 + 4k_1^2)^{1/2}$. It is interesting that for any combination of waves determined by the weight factor θ, the second order (α^2) contribution does not depend on the drift direction coordinate y. Thus, in the second order pure zonal flow is generated, the corresponding ion velocity components are $v_x = - \partial\Phi_2/\partial y = 0$ and

$$v_y = \partial\Phi_2/\partial x = (k_1^2 k_2/8) \cos(2k_1 x) \cos(2k_3 z)\, (2(\omega_1 + \omega_2 + (\omega_1 - \omega_2) \cos(2\theta) \sin^2(\omega_3 t/2). \qquad (8)$$

In this way the result obtained in [5] for the particular case $\theta = 0$ is generalized to the temporal evolution of the arbitrary combination of the waves with the frequencies (in the linear approximation) ω_1 and ω_2. This solution describes zonal flow generation by the combination of initially monochromatic drift ion acoustic waves. The generation is self-consistent, instead of the zonal flow generation by the drift-wave pump is considered in [6].

In the third order (α^3) the shifts $\delta\omega_{11}$ and $\delta\omega_{12}$ of the main frequency ω_1 appear in Φ_1 and v_1, respectively ($\theta = 0$ expressions are presented for simplicity):

$$(\delta\omega_{11} / \omega_1) = (\, \alpha^2 /16\,)\, k_1^2\, k_2^2\, (k_2^2 - 3\, k_1^2) / (1 + k_1^2 + k_2^2),$$

$$(9)$$

$$(\delta\omega_{12} / \omega_1) = - (\, \alpha^2 /16\,)\, k_1^2\, k_2^2.$$

The presence of the terms with combination frequencies in the third order can lead to the parametric instabilities.

In conclusion, it must be noted that the 3D model (1) includes many physical effects: higher harmonics generation and frequency shifts, parametric instabilities, zonal flow generation by initially monochromatic waves etc. Symmetry considerations facilitate essentially their treatment by helping us to obtain exact, perturbative and numerical solutions of the nonlinear problems.

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ДРЕЙФОВО ІОННОАКУСТИЧНІ ХВИЛІ ТА ЇХНІ СИМЕТРІЇ

В. Б. Таранов

Розглядається тривимірна модель для пов'язаних між собою дрейфових та іонноакустичних хвиль. Отримані перетворення симетрії цієї моделі в присутності магнітного ширу та у безшировому випадку. Знайдено деякі з найбільш симетричних розв'язків, точних та одержаних за допомогою теорії збурень. Зокрема, одержані розв'язки, що представляють генерацію зональних потоків початково монохроматичними хвилями.

ДРЕЙФОВО ИОННОАКУСТИЧЕСКИЕ ВОЛНЫ И ИХ СИММЕТРИИ

В. Б. Таранов

Рассмотрена трехмерная модель для связанных дрейфовых и ионноакустических волн. Получены преобразования симметрии этой модели в присутствии магнитного шира и в безшировом случае. Найдены некоторые из наиболее симметричных решений, точных и полученных с помощью теории возмущений. В частности, получены решения, представляющие генерацию зональных потоков первоначально монохроматическими волнами.

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