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SYMMETRIES OF THE KINETIC PLASMA THEORY

V. B. Taranov

Institute for Nuclear Research, Academy of Sciences of Ukraine, Kyiv

Symmetry transformations are found for the kinetic theory of the upper hybrid oscillations of the electron plasma. It is shown that in the cold electron plasma limit the symmetry extension takes place, allowing us to obtain the general solution which is analogous to that found previously in the Lagrangian variables. The results are compared to the known symmetry properties of the Langmuir oscillations of the electron plasma. The algorithm used in the present work to obtain the symmetries of kinetic models of the plasma theory is illustrated on the example of Langmuir oscillations in the multi-component plasma.

1. Introduction

In recent decades the Lie group method has been applied to explore many physically interesting nonlinear problems in gas dynamics, plasma physics etc. [1]. Furthermore, extensions of the classical Lie algorithm to the integro-differential systems of equations of kinetic theory were proposed [1 - 4].

In this work, the results obtained previously [2, 3] for the electron plasma high frequency longitudinal waves in the absence of an external magnetic field are generalized to the case when the constant external magnetic field is present, i.e. for the upper hybrid waves. In the Section 2, the corresponding nonlinear integro-differential model equations are derived. In the Section 3, the symmetry transformations obtained for the upper hybrid waves are presented, together with their extension in the cold electron plasma case. For the sake of completeness, similar previous results for the electron Langmuir waves are presented in the Section 4. Langmuir waves in the multi-component plasma are considered in the Section 5. An algorithm used for the investigation of the kinetic theory integro-differential equations is briefly reviewed. Lie group of symmetry is obtained and its extension for the plasma containing the components with equal or closed values of the charge to mass ratio of particles is emphasized. Conclusions are made in the Section 6.

2. Upper hybrid waves model

Let us consider high frequency plasma motion with constant ion background density n_0. In this case Vlasov - Maxwell integro-differential system of equations holds:

$$\partial f/\partial t + \mathbf{v} \cdot \partial f/\partial \mathbf{r} - (e/m)[\mathbf{E} + (1/c)(\mathbf{v} \times \mathbf{B})] \cdot \partial f/\partial \mathbf{v} = 0,$$

$$\nabla \times \mathbf{E} + (1/c)\, \partial \mathbf{B}/\partial t = 0, \quad \nabla \cdot \mathbf{E} = 4\pi e\, (n_0 - \textstyle\int f\, d\mathbf{v}), \tag{1}$$

$$\nabla \times \mathbf{B} = (1/c)\, \partial \mathbf{E}/\partial t - (4\pi e/c) \textstyle\int \mathbf{v} f\, d\mathbf{v}, \quad \nabla \cdot \mathbf{B} = 0$$

where f(t, **r**, **v**) is electron distribution function. Let us assume that the plasma is subjected to a constant external magnetic field

$$\mathbf{B} = B_0\, \mathbf{e}_z, \qquad B_0 = \text{const},$$

so the electron cyclotron frequency is equal to $\omega_{ce} = eB_0 / mc$. Let us restrict ourselves to the transverse plasma movements (see, e.g. [5]):

$$\mathbf{E} = E(t,x)\, \mathbf{e}_x, \qquad \partial/\partial z = 0, \qquad \partial/\partial y = 0.$$

Electron distribution function f has been integrated over v_z and longitudinal current is not present, $\int v_z f\, dv_z = 0$. In this way we obtain the simplified system of equations describing upper hybrid waves in the electron plasma:

$$\partial f/\partial t + v_x\, \partial f/\partial x - (e/m)\, E\, \partial f/\partial v_x + \omega_{ce}\, (v_x\, \partial f/\partial v_y - v_y\, \partial f/\partial v_x) = 0,$$

$$\partial E/\partial x = 4\pi e\, (n_0 - \int f\, d\mathbf{v}), \qquad (2)$$

$$\partial E/\partial t = 4\pi e \int v_x\, f\, d\mathbf{v},$$

where $d\mathbf{v} = dv_x dv_y$. This simplified system, however, still remains integro-differential. External current density must be added to avoid possible nonlinear generation of the magnetic field:

$$\mathbf{j}_0 = j_0\, \mathbf{e}_y\,, \quad j_0 = e \int v_y\, f\, d\mathbf{v}.$$

This statement completes the formulation of the upper hybrid model equations.

3. Symmetries and solutions for the upper hybrid waves

Continuous symmetry transformations for the integro-differential system (2) can be found having used an indirect method exploring the symmetry of an infinite set of equations for the moments of the distribution function f [2, 3]. After some complicated but straightforward algebra, we obtain the following Lie symmetry group admitted by the system (2):

$$X_1 = \partial/\partial t, \quad X_2 = \partial/\partial x,$$

$$X_3 = \partial/\partial v_y - (m/e)\, \omega_{ce}\, \partial/\partial E,$$

$$X_4 = x\, \partial/\partial x + E\, \partial/\partial E + v_x\, \partial/\partial v_x + v_y\, \partial/\partial v_y - 2f\, \partial/\partial f, \qquad (3)$$

$$X_5 = \cos(\omega_{uh}t)\, \partial/\partial x - \omega_{uh} \sin(\omega_{uh}t)\, \partial/\partial v_x + \omega_{ce} \cos(\omega_{uh}t)\, \partial/\partial v_y + (m/e)\, \omega_{uh}^2 \cos(\omega_{uh}t)\, \partial/\partial E,$$

$$X_6 = \sin(\omega_{uh}t)\, \partial/\partial x + \omega_{uh} \cos(\omega_{uh}t)\, \partial/\partial v_x + \omega_{ce} \sin(\omega_{uh}t)\, \partial/\partial v_y + (m/e)\, \omega_{uh}^2 \sin(\omega_{uh}t)\, \partial/\partial E,$$

where $\omega_{pe}^2 = 4\pi e^2 n_0/m$ is electron Langmuir frequency and $\omega_{uh}^2 = \omega_{pe}^2 + \omega_{ce}^2$ is the frequency of the upper hybrid oscillations.

A considerable extension of the symmetry takes place in the cold electron plasma limit, i.e. for the distribution functions of the form

$$f(t, x, v_x, v_y) \equiv n(t, x)\, \delta(v_x - u(t, x))\, \delta(v_y - v(t, x)). \qquad (4)$$

Equations (2) are reduced to the partial differential equations for the functions u, v and E of the variables t and x:

$$\partial u/\partial \tau = -(e/m)E - \omega_{ce}v, \quad \partial v/\partial \tau = \omega_{ce}u, \quad \partial E/\partial \tau = 4\pi e n_0 u \qquad (5)$$

where $\partial/\partial \tau = \partial/\partial t + u\, \partial/\partial x$. Their solution by the use of Lagrangian variables is made possible due to adding to (5) the equation for $x(\tau)$:

$$\partial x/\partial \tau = u \qquad (6)$$

and treating x, E, u and v as the functions of τ and the initial value of $x(\tau)$, i.e. $x_0 = x(0)$. It is very easy to observe that:

$$\partial^2 u/\partial \tau^2 + \omega_{uh}^2\, u = 0$$

and the general solution is as follows:

$$x = I_1 \cos(\omega_{uh}\tau) + I_2 \sin(\omega_{uh}\tau) + I_3$$

$$u = \omega_{uh}\, [-I_1 \sin(\omega_{uh}\tau) + I_2 \cos(\omega_{uh}\tau)] \qquad (7)$$

$$E = 4\pi e n_0 \,[I_1 \cos(\omega_{uh}\tau) + I_2 \sin(\omega_{uh}\tau) + I_4]$$

$$v = \omega_{ce} \,[I_1 \cos(\omega_{uh}\tau) + I_2 \sin(\omega_{uh}\tau)] - (\omega_{pe}^2/\omega_{ce})\, I_4$$

where $\mathbf{I} = \{ I_1, \dots, I_4 \}$ are the functions of x_0 which are determined by the initial conditions, e.g. $I_2(x_0) = u(0, x_0) / \omega_{uh}$. In this way complicated expressions for x, E, u and v presented in [6] can be obtained. From the point of view of the symmetry approach, this possibility of finding the general solution in Lagrangian variables is related to the fact that the system (5), (6) can be presented in a form

$$\partial \mathbf{I}/\partial \tau = 0 \tag{8}$$

which is invariant, as it is readily seen, under the wide class of transformations depending on arbitrary functions **F** and G:

$$\mathbf{I}' = \mathbf{F}(\mathbf{I}), \quad \tau' = G(\tau, \mathbf{I}). \tag{9}$$

The transformations (9) allow us to obtain the general solution of the system (5), (6) starting from the trivial zero solution. In fact, the trivial solution

$$I_2 = 0, \dots , I_4 = 0$$

is generalized by the transformation (9) to

$$F_2(I_1, \dots , I_4) = 0, \dots , F_4(I_1, \dots , I_4) = 0,$$

with arbitrary functions $F_2 , \dots , F_4$. Then we can express the solution in a form

$$I_2 = g_2 (I_1), \dots , I_4 = g_4 (I_1).$$

Finally, we determine the functions g_2 to g_4 by the general initial conditions for u, v and E. This will reproduce the solution obtained in Lagrangian variables. So the possibility of solving cold electron plasma equations is due to the sufficiently large extension of the model symmetry.

4. Langmuir electron plasma waves

Let us review shortly the previous results obtained in [2, 3] for the longitudinal electron plasma waves. In this case, i.e. for the system of equations (see, e.g. [6]):

$$\begin{gathered} \partial f/\partial t + v_z\, \partial f/\partial z - (e/m)\, E_z\, \partial f/\partial v_z = 0, \\ \partial E_z/\partial z = 4\pi e\, (n_0 - \textstyle\int f\, dv_z), \\ \partial E_z/\partial t = 4\pi e \textstyle\int v_z\, f\, dv_z) \end{gathered} \tag{10}$$

where the distribution function f has been integrated over v_x and v_y, the symmetry group is as follows [2, 3]:

$$\begin{gathered} X_1 = \partial/\partial t, \quad X_2 = \partial/\partial z, \\ X_3 = \cos(\omega_{pe}t)\, \partial/\partial z - \omega_{pe} \sin(\omega_{pe}t)\, \partial/\partial v_z + (m/e)\, \omega_{pe}^2 \cos(\omega_{pe}t)\, \partial/\partial E_z, \\ X_4 = \sin(\omega_{pe}t)\, \partial/\partial z + \omega_{pe} \cos(\omega_{pe}t)\, \partial/\partial v_z + (m/e)\, \omega_{pe}^2 \sin(\omega_{pe}t)\, \partial/\partial E_z, \\ X_5 = z\, \partial/\partial z + E_z\, \partial/\partial E_z + v_z\, \partial/\partial v_z - f\, \partial/\partial f. \end{gathered} \tag{11}$$

In the cold electron plasma limit

$$f(t, z, v_z) \equiv n(t, z)\, \delta(v_z - w(t, z)) \tag{12}$$

the system (6) is reduced to the partial differential equations

$$\partial w/\partial \tau = -(e/m)E_z, \quad \partial E_z/\partial \tau = 4\pi e n_0 w \tag{13}$$

where $\partial/\partial\tau = \partial/\partial t + w\, \partial/\partial z$. Invariants $\mathbf{I} = \{ I_1, I_2, I_3)$ in this case have the form

$$I_1 = z - E_z / 4\pi e n_0,$$

$$I_2 = (w/\omega_{pe}) \cos(\omega_{pe}\tau) + (E_z / 4\pi e n_0) \sin(\omega_{pe}\tau), \tag{14}$$

$$I_3 = (w/\omega_{pe}) \sin(\omega_{pe}\tau) - (E_z / 4\pi e n_0) \cos(\omega_{pe}\tau).$$

The invariance under the transformations (9) made it possible to obtain the general solution of (13) from the trivial one or to solve this system in Lagrangian variables as it was done in [6].

5. Langmuir waves in the multi-component plasma

Let us consider now high frequency oscillations of the multi-component plasma described by the Vlasov - Maxwell equations for the distribution functions $f_\alpha(t, x, v)$, where the index $\alpha = 1, \dots, N$ is used to label plasma components, and the electric field $E(t, x)$:

$$\partial f_\alpha/\partial t + v\, \partial f_\alpha/\partial x + (e_\alpha/m_\alpha)\, E\, \partial f_\alpha/\partial v = 0$$

$$\partial E/\partial x = 4\pi\, \Sigma_\alpha\, e_\alpha \int f_\alpha\, dv \tag{15}$$

and

$$\partial E/\partial t + 4\pi\, \Sigma_\alpha\, e_\alpha \int v\, f_\alpha\, dv = 0 \tag{16}$$

where Σ_α means the sum over α from 1 to N, charge and mass of particles of the α-th component of the plasma are denoted by e_α and m_α, respectively.

Let us now review shortly the algorithm [3] which will allow us to obtain the symmetry group of the system (15). First, we introduce the moments of distribution functions:

$$M_{k,\alpha}(t, x) \equiv \int v^k f_\alpha(t, x, v)\, dv \tag{17}$$

where $k = 0, 1, \dots$. From the system (15) we readily obtain the partial differential equations for the moments $M_{k,\alpha}$ and the electric field E:

$$\partial M_{k,\alpha}/\partial t + \partial M_{k+1,\alpha}/\partial x - (e_\alpha/m_\alpha)\, E\, k\, M_{k-1,\alpha} = 0,$$

$$\partial E/\partial x = 4\pi\, \Sigma_\alpha\, e_\alpha\, M_{0,\alpha}. \tag{18}$$

Then, we restrict ourselves to the truncated system of equations (18) with $k = 0, 1, \dots, k_{max}$, so the continuous symmetry group in the space of the variables t, x, E, $M_{k,\alpha}$ can be found by the standard Lie method. This group is as follows [3]:

$$X_1 = \partial/\partial t, \qquad X_2 = \partial/\partial x,$$

$$X_3 = t\, \partial/\partial x + \Sigma_{k,\alpha}\, k\, M_{k-1,\alpha}\, \partial/\partial M_{k,\alpha}$$

$$X_4 = t\, \partial/\partial t - 2\, E\, \partial/\partial E - \Sigma_{k,\alpha}\, (k + 2)\, M_{k,\alpha}\, \partial/\partial M_{k,\alpha} \tag{19}$$

$$X_5 = x\, \partial/\partial x + E\, \partial/\partial E - \Sigma_{k,\alpha}\, k\, M_{k,\alpha}\, \partial/\partial M_{k,\alpha}$$

$$X_\Psi = \Sigma_\alpha\, (\, - \partial\Psi_\alpha/\partial x\, \partial/\partial M_{kmax,\alpha} + \partial\Psi_\alpha/\partial t\, \partial/\partial M_{kmax+1,\alpha}$$

where $\Psi_\alpha(t, x) = A_\alpha(t)\, x + B_\alpha(t)$, A_α and B_α are arbitrary functions of t.

The transformation generated by X_Ψ acts on the moments $M_{kmax,\alpha}$ and $M_{kmax+1,\alpha}$ alone. So it disappears in the limit $k_{max} \to \infty$. To obtain the rest of transformations X_1 - X_5 in the kinetic variables t, x, v, E and f_α the generating functional F_α can be used:

$$F_\alpha (t, x \mid \kappa) \equiv \int f_\alpha(t, x, v)\, e^{i\kappa v}\, dv = \Sigma_n ((i\kappa)^n/n!)\, M_{n,\alpha}(t, x). \quad (20)$$

In this way, we finally obtain (see [3]) the following Lie group of symmetry for the system (15):

$$X_1 = \partial/\partial t, \quad X_2 = \partial/\partial x,$$

$$X_3 = t\, \partial/\partial x + \partial/\partial v,$$

$$X_4 = t\, \partial/\partial t - 2\, E\, \partial/\partial E - v\, \partial/\partial v - \Sigma_\alpha f_\alpha\, \partial/\partial f_\alpha, \quad (21)$$

$$X_5 = x\, \partial/\partial x + E\, \partial/\partial E + v\, \partial/\partial v - \Sigma_\alpha f_\alpha\, \partial/\partial f_\alpha.$$

In addition, if there are the plasma components with

$$(e_\mu/m_\mu) = (e_\nu/m_\nu) \quad (22)$$

for some μ and ν, additional symmetries appear

$$X_{\mu\nu} = f_\mu\, (\partial/\partial f_\mu - (e_\mu/e_\nu)\, \partial/\partial f_\nu). \quad (23)$$

The property (22) is satisfied (at least, it is good approximation), for deuterium and helium ions.

The same symmetry group can be found [3] by the same algorithm if we consider system of the equations (15), (16) instead of the pure Vlasov-Poisson system (15).

It must be noted that the symmetry extension (23) takes place not only for the Langmuir waves described by the simplified system of equations (15) or (15), (16), but also in the general three dimensional case:

$$\partial f_\alpha/\partial t + \mathbf{v} \cdot \partial f_\alpha/\partial \mathbf{r} + (e_\alpha/m_\alpha)[\mathbf{E} + (1/c)(\mathbf{v} \times \mathbf{B})] \cdot \partial f_\alpha/\partial \mathbf{v} = 0,$$

$$\nabla \times \mathbf{E} + (1/c)\, \partial \mathbf{B}/\partial t = 0, \quad \nabla \cdot \mathbf{E} = 4\pi\, \Sigma_\alpha e_\alpha \int f_\alpha\, d\mathbf{v}, \quad (24)$$

$$\nabla \times \mathbf{B} = (1/c)\, \partial \mathbf{E}/\partial t - (4\pi/c)\, \Sigma_\alpha e_\alpha \int \mathbf{v} f_\alpha\, d\mathbf{v}, \quad \nabla \cdot \mathbf{B} = 0.$$

Thus, this symmetry extension is characteristic for the general collisionless plasma kinetic theory.

6. Conclusions

In this work the invariance of the integro-differential equations for the nonlinear upper hybrid waves (2) was studied. Continuous symmetry group (3) has been obtained. Among the symmetries are time and space homogeneity (X_1 and X_2), the non relativistic remnant of the Lorentz transform in the direction perpendicular to the external magnetic and the perturbed electric fields (X_3), similarity transform (X_4) related to the fact that in the equations (2) no assumptions are made *a priori* about the plasma temperature and thus no characteristic thermal velocity is contained in the system. Transformations X_5 and X_6 are the peculiar specific ones for the upper hybrid waves. They mean that the spatially homogeneous upper hybrid oscillations can be included in arbitrary solution of the model as, e.g. the nonlinear reaction of the system to the rapid homogeneous external current flash.

It was shown also that in the cold electron plasma limit (4) the symmetry extension allows us to obtain implicit, but general solution, which is equivalent to better known procedure of solving equations in Lagrangian variables. Comparison of the above mentioned results with symmetries and solutions for more simpler theory of electron plasma high frequency waves in the absence of the external magnetic field shows very close qualitative analogy.

An algorithm for the investigation of the kinetic plasma theory symmetries is presented and illustrated on the multi-component plasma model for the Langmuir waves. This algorithm, firstly published in the preprint [3], was used in the present work for the 1D in x-space and 2D in v-space model of upper hybrid waves.

Symmetry extension for the plasma containing the components with closed values of the charge to mass ratio of particles was emphasized, following the preprint [3]. This property is characteristic for the general collisionless kinetic plasma theory (24).

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СИМЕТРІЇ КІНЕТИЧНОЇ ТЕОРІЇ ПЛАЗМИ

В. Б. Таранов

Одержано перетворення симетрії для кінетичної теорії верхньогібридних коливань електронної плазми. Показано, що у наближенні холодної електронної плазми відбувається значне розширення симетрії, завдяки якому стає можливим одержання загального розв'язку, аналогічного до знайденого раніше при застосуванні змінних Лагранжа. Результати порівняно з відомими властивостями симетрії ленгмюрівських коливань електронної плазми. Застосований у роботі для знаходження симетрій кінетичних моделей теорії плазми алгоритм проілюстровано на прикладі ленгмюрівських коливань багатокомпонентної плазми.

СИММЕТРИИ КИНЕТИЧЕСКОЙ ТЕОРИИ ПЛАЗМЫ

В. Б. Таранов

Найдены преобразования симметрии кинетической теории верхнегибридных колебаний электронной плазмы. Показано, что в приближении холодной электронной плазмы происходит значительное расширение симметрии, благодаря которому становится возможным получение общего решения, аналогичного найденному ранее с помощью переменных Лагранжа. Результаты сопоставлены с известными свойствами симметрии ленгмюровских колебаний электронной плазмы. Примененный в работе для получения симметрий кинетических моделей теории плазмы алгоритм проиллюстрирован на примере ленгмюровских колебаний многокомпонентной плазмы.

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