SMOOTH TRAJECTORIES ON TOROIDAL MANIFOLDS

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Great advance in plasma magnetic confinement theory is attained by means of trajectories investigation on toroidal manifolds. Since local and global aspects of qualitative trajectories flow are important at that, it is natural to consider such trajectories on smooth manifolds. It is succeeded by means of fixing the local metrics of a manifold to integrate equation for smooth trajectories, to find out connection between infinitesimal and topology trajectory properties, to write down equations of marked curves in evident form. Conditions are given, on which smooth trajectories will be either closed or compact on a manifold. Restrictions of topological invariances are found for loxodromies, for which a trajectory will be a plane curve.

1. For the topology of differential manifolds an initial point is an elementary theory of smooth manifolds. These topological ideas and methods have been widely adopted in various branches of theoretical physics. The connection between topology and differential geometry is settled by "global" geometry, which has the goal to give some information about space topology by means of local measurements carried out everywhere in this space. An idea consists in the fact that on many three-dimensional manifolds one can put in "good" metrics, which let to obtain new more thorough understanding properties of these manifolds. Since, the local and global features are important simultaneously at that it is naturally to consider such metrics on the smooth manifolds.

In the present paper it was chosen such toroidal space coordinates, which permit to get the evident formulas for the loxodromies of the torus, geodesic trajectories and investigate these properties.

2. It is considered toroidal coordinates [1] by means of fixing links with the point Cartesian coordinates

$$x = \frac{csh\eta\cos\varphi}{ch\eta - \cos\theta};\quad y = \frac{csh\eta\sin\varphi}{ch\eta - \cos\theta};\quad z = \frac{c\sin\theta}{ch\eta - \cos\theta}. \tag{1}$$

The scale factor

$$c = \sqrt{R^2 - a^2} \tag{2}$$

is the length of tangent from the coordinates origin to a tore in the plane φ=const, R-the toroidal axis radius (azimuth), a – meridian radius of a tore.

The coordinate surfaces are: η=const

$$\left(\sqrt{x^2 + y^2} - ccth\eta\right)^2 + z^2 = \frac{c^2}{sh^2\eta}\quad - \tag{3}$$

toroidal manifolds; θ=const

$$x^2 + y^2 + (z - cctg\,\theta)^2 = \frac{c^2}{\sin^2\theta}\quad - \tag{4}$$

spheres, centers of which have in the Cartesian system (0,0,c×ctgθ); φ=const – planes containing the z axis.

Having chosen $\eta = \eta_0$ in equation (3), we confine attention on the case where

$$\frac{c}{sh\eta_0} = a, \quad c \times cth\eta_0 = R. \tag{5}$$

The values $0 < \eta < \eta_0$ correspond to the external region of a torus, the values $\eta > \eta_0$ - to the internal region of a torus. The internal toroidal region contains an azimuth axis of a tore $\eta \to \infty \left(z = 0,\ x^2 + y^2 = R^2\right)$.

At round of a tore θ is changed on 2π; that is why a single – valued function of a point in space will be periodic one of θ with period equal 2π.

3. Let a point lying on a toroidal surface is rotating uniformly around the toroidal axis so that

$$\varphi = \lambda(t - t_0) \tag{6}$$

and around the azimuth axis

$$\theta = \varsigma(t - t_0). \tag{7}$$

Substitution t from equation (6) to equation (7) yields linear dependence between angles:

$$\varphi = \frac{\lambda}{\varsigma}\theta. \tag{8}$$

At the irrational proportionality factor trajectory (8) is compact on the torus surface. An angle between toroidal meridian and trajectory (8)

$$\cos\alpha = \frac{\varsigma}{\sqrt{\varsigma^2 + \lambda^2 sh^2\eta_0}} \tag{9}$$

does not depend on coordinates of point. One can say in another words, trajectory (8) crosses meridian under the steady angle. Such trajectories are named as loxodromies [2].

If trajectory (8) closes itself after m turns around torus axis and n turns around azimuth axis then the equation of periodic loxodromy has the form

$$\varphi = \frac{m}{n}\theta. \tag{10}$$

As far as the ratio m and n is a rational value then the closed loxodromy flows as in the region of a toroidal surface, where the Gauss curvature is positive so as and in the regions, where it is negative, i. e. (10) is a space curve.

At m=n=1 the closed loxodromy will be a plane curve [3]; normal to this plane has the vector part

$$\vec{N}(\theta,\varphi) = \frac{1}{ch\eta_0 - \cos\theta}\left((ch\eta_0\cos\theta - 1)\cos\varphi, (ch\eta_0\cos\theta - 1)\sin\varphi, -sh\eta_0\sin\theta\right). \tag{11}$$

Reference curve is a circle with the radius $c \times cth\eta_0$ and the center in the point $(c/sh\eta_0, 0, 0)$.

For the periodic loxodromy an angle between meridian and trajectory

$$tg\alpha = \frac{m}{n} sh\eta_0 \tag{12}$$

is proportional to the ratio of rotation numbers around toroidal axes.

4. Geodesic lines represent itself trajectories of natural movement. Importance of geodesics is caused by the fact that the geodesic trajectories theory can be constructed in an analytical form.

Accordingly with (1) tore is a surface of rotation that is why integrals of geodesics will be written in an evident form.

In the theory of geodesics two methods are existed. One of them is the use of variation principle. As far as relations (1) are determined space toroidal metrics at the same time Lagrangian can be written in toroidal coordinates. Geodesic trajectory is minimized the particle energy.

In geometrical method geodesic is treated as a line on the surface, projection of curvature vector of which in a point on the tangent plane to the surface is equal zero. The set of equations, determined geodesics, we wrote in natural form [4]:

$$\frac{d\theta}{ds}=\frac{ch\eta_0-\cos\theta}{c}\cos\alpha;\quad \frac{d\varphi}{ds}=\frac{ch\eta_0-\cos\theta}{c\cdot sh\eta_0}\sin\alpha;\quad \frac{d\alpha}{ds}=\frac{\sin\theta}{c}\sin\alpha. \tag{13}$$

The first integral of equations (13)

$$\sin\alpha=h(ch\eta_0-\cos\theta) \tag{14}$$

is composed the maintenance of the Clerot theorem and links the angle α between meridian and geodesic with the point coordinates on the toroidal surface. The angle α is minimal at crossing by geodesic the external toroidal equator and reach maximum going on the internal equator. Therefore, Clerot constant h has a limited change $(ch\eta_0-1)^{-1}\le h\le(ch\eta_0+1)^{-1}$. If Clerot constant does not satisfy the marked inequality, then a geodesic trajectory is in a restrict surface region of a tore.

The second integral of equations (13)

$$\varphi=\frac{h}{sh\eta_0}\int\frac{(ch\eta_0-\cos\theta)d\theta}{\sqrt{1-h^2(ch\eta_0-\cos\theta)^2}} \tag{15}$$

is finite with all derivatives at θ=0 and hence in the whole space. When the Clerot theorem (14) is valid, φ is a continuous function of angle θ and geodesic (15) coats tightly toroidal surface. To derive analytical representation of geodesic trajectory flow we consider the case when Clerot constant is small. Let us expand (15) in a series on the power of small quantity h:

$$\varphi\approx\frac{h}{sh\eta_0}\left\{\theta ch\eta_0-\sin\theta+\frac{h^2}{2}\left[\left(ch^3\eta_0+\frac{3}{2}ch\eta_0\right)\theta-3\left(ch^2\eta_0+\frac{1}{4}\right)\sin\theta+\right.\right.$$
$$\left.\left.+\frac{3}{4}ch\,\eta_0\sin 2\theta-\frac{1}{12}\sin 3\theta\right]\right\}. \tag{16}$$

The approximate expression (16) shows, that φ changes a little at the chance θ. For clearing geometrical meaning of smallness of Clerot constant let us consider a geodesic which is closing after p rounds around the toroidal axis and q rounds around the azimuth axis. This condition let to write down Clerot constant in a form:

$$h\approx\frac{p}{q}th\eta_0-\left(\frac{p}{q}\right)^3\frac{th^3\eta_0}{2}\left(ch^2\eta_0+\frac{3}{2}\right)-\dots\,. \tag{17}$$

Then the smallness of Clerot constant means that for the periodic geodesic trajectory

$$\varphi\approx\frac{p}{q}\left\{\theta-\frac{1}{ch\eta_0}\sin\theta-\left(\frac{p}{q}\right)^2\frac{th^2\eta_0}{ch\eta_0}\left[\left(ch^2\eta_0-\frac{3}{8}\right)\sin\theta-\frac{3}{8}ch\eta_0\sin 2\theta+\frac{1}{24}\sin 3\theta\right]\right\} \tag{18}$$

the number of rounds around the tore axis is much greater than the number of crossings by trajectory external (internal) equator.

As it can be seen from comparison (10) and (18), the closed loxodromy turns faster around the tore axis than the geodesic at the other equal conditions.

Let us evaluate further the integral (15) when the angle between outer equator and geodesic trajectory is small

$$h \approx (ch\eta_0 + 1)^{-1} - \varepsilon. \tag{19}$$

An asymptotic evaluation

$$\varphi \approx \frac{1}{sh\eta_0}\left\{-2\ln\left(\sqrt{ch\eta_0 + \sin^2\frac{\theta}{2}} + \sin\frac{\theta}{2}\right) + \ln ch\eta_0 + \sqrt{ch\eta_0 + 1}F(\varphi^+, \kappa)\right\} \tag{20}$$

is valid at the implication of the inequality $\varepsilon(ch\eta_0 + 1)\langle\langle 1$. Elliptic function $F(\varphi^+, \kappa)$ has an argument $\varphi^+ \approx arctg\left(\sqrt{\frac{ch\eta_0 + 1}{ch\eta_0}}tg\frac{\theta}{2}\right)$ and square of the additional module $k'^2 \approx (\varepsilon/2)ch\eta_0(ch\eta_0 + 1)$.

Condition of geodesic trajectory closeness let to evaluate the value ε. Calculations leads to result

$$\varepsilon \approx \frac{32}{(ch\eta_0 + 1)ch\eta_0}\exp\left\{-\frac{2}{\sqrt{ch\eta_0 + 1}}\left[\pi\frac{p}{q}sh\eta_0 + 2\ln\frac{\sqrt{ch\eta_0 + 1}}{\sqrt{ch\eta_0}}\right]\right\}. \tag{21}$$

Exponential index at any η_0 is a negative value. Hence, above - mentioned inequality is valid.

Using (21), analytical representation of geodesic trajectory can be written in the form

$$\varphi \approx \frac{2}{sh\eta_0}\left\{-\ln\left(\sqrt{ch\eta_0 + \sin^2\frac{\theta}{2}} + \sin\frac{\theta}{2}\right) + \frac{1}{2}\ln ch\eta_0 + \right.$$
$$\left. + \left(\frac{p}{q}sh\eta_0 + \frac{2}{\pi}\ln\frac{\sqrt{ch\eta_0 + 1} + 1}{\sqrt{ch\eta_0}}\right)arctg\left(\sqrt{\frac{ch\eta_0 + 1}{ch\eta_0}}tg\frac{\theta}{2}\right)\right\}. \tag{22}$$

Representation (22) is a finite one at the crossing by geodesic internal and external equators and describes the flow of geodesic on the circular cross-section toroidal regions where the Gauss curvature is as positive so as negative.

5. In the present paper we studied the behaviour of trajectories directed by equations (8) and (15) on the tore of circular cross - section. It is manifested that the trajectory of uniform rotation around toroidal axes is a loxodromy. This has been achieved by means of investigating the angle between trajectory and meridian of a tore. We find that the closed trajectory with periods equal unity is a circle of $c \times cth\eta_0$, radius lying in the plane with a the normal (11).

For the geodesic trajectories it is shown that the line (16) will be coated tightly the toroidal surface if Clerot constant has a limited change $(ch\eta_0 - 1)^{-1} << h << (ch\eta_0 + 1)^{-1}$.

For the closed geodesic trajectories it is possible the analytical representation (18), when the trajectory pitch angle to the torus meridian is small. It means geometrically that number rounds of line around the torus axis up to closing is greater than number of rounds around the azimuth axis.

In the case when the angle between trajectory and external equator is small asymptotic representation of geodesic (22) is finite and at the crossing by the line of internal (external) equator. The crossing number is an arbitrary value.

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ГЛАДКІ ТРАЄКТОРІЇ НА ТОРОЇДАЛЬНИХ МНОГОВИДАХ

С. С. Романов

Великих успіхів досягнуто в теорії магнітного утримання плазми шляхом вивчення траєкторій на тороїдальних многовидах. Оскільки при цьому є важливими локальні та глобальні аспекти якісних потоків траєкторій, природно розглянути такі траєкторії на гладких многовидах. Це досягається заданням локальних метрик многовидів для інтегрування рівняння гладких траєкторій, знаходження зв'язку між інфінітезимальними та топологічними властивостями траєкторій та написання рівнянь вибраних кривих в простій формі. Наведено умови, за яких гладкі траєкторії будуть замкненими або компактними на многовиді. Знайдено обмеження на топологічні інваріанти для локсодромій, для яких траєкторією буде плоска крива.