EVALUATION OF AN EFFECTIVE RIPPLE IN STELLARATORS [*]

V. V. Nemov [1], S. V. Kasilov [1], W. Kernbichler [2], M. F. Heyn [2]

[1] *Institute of Plasma Physics, National Scientific Center "Kharkov Institute of Physics and Technology",
Kharkov, Ukraine*
[2] *Institut für Theoretische Physik, Graz, Austria*

Diffusion coefficients and peculiarities of the different representations of ε_{eff} (effective ripple) are discussed.

Introduction

An effective ripple, ϵ_{eff}, obtained from neoclassical transport computations in the $1/\nu$ regime is often used for evaluating the confinement properties of stellarator systems. In the $1/\nu$ regime, the ϵ_{eff} value enters the expression for transport coefficients as a factor $\epsilon_{eff}^{3/2}$. It can be effectively used as a target function within codes for stellarator optimization. There exist different possibilities for a representation of the ϵ_{eff} parameter and, in addition, there are problems to determine the set of quantities which is sufficient for computing ϵ_{eff} in Boozer coordinates. In particular, these points are important for transport codes which evaluate the quantity ϵ_{eff} and for subsequent computation of particle (and heat) fluxes or diffusion coefficients.

General expression for ϵ_{eff}

For an arbitrary stellarator magnetic field, the particle flux density F_n in the $1/\nu$ transport regime is linked with the conventional stellarator flux density $F_n^{(conv)}$ through the relation [1]

$$F_n = \frac{\epsilon_{eff}^{3/2}}{\epsilon_h^{3/2}} F_n^{(conv)}, \quad F_n^{(conv)} = -\frac{\sqrt{8}}{9\pi^{3/2}} \frac{v_T^2 \rho_L^2}{\nu R^2} \epsilon_h^{3/2} \int\limits_0^\infty \frac{dz\, e^{-z} z^{5/2}}{A(z)} \frac{n}{f_m} \frac{\partial f_m}{\partial r}. \tag{1}$$

Here, ϵ_{eff} and ϵ_h are the effective ripple and the helical ripple along the magnetic field line, respectively, $f_m = f_m(\psi, w)$ is the Maxwellian distribution, ψ is the magnetic surface label, and R is the major radius of the torus. All other quantities are the same as in Ref.[1]. The derivative of f_m is taken with respect to a formal radius of the magnetic surface, r, which corresponds to the definition

$$\frac{\partial f_m}{\partial r} = \frac{\partial f_m}{\partial \psi} \langle |\nabla\psi| \rangle, \quad \langle |\nabla\psi| \rangle = \lim_{L_s \to \infty} \left(\int\limits_0^{L_s} \frac{ds}{B} \right)^{-1} \int\limits_0^{L_s} \frac{ds}{B} |\nabla\psi|. \tag{2}$$

The quantity ϵ_{eff} is obtained in [1] for an arbitrary stellarator magnetic field taking into account all classes of trapped particles. For convenience in the further discussion, it is presented as

$$\epsilon_{eff}^{3/2} = \hat{\epsilon}_{eff}^{3/2} / \langle |\nabla\psi| \rangle^2, \tag{3}$$

with

$$\hat{\epsilon}_{eff}^{3/2} = \frac{\pi R^2}{8\sqrt{2}} \lim_{L_s \to \infty} \left(\int\limits_0^{L_s} \frac{ds}{B} \right)^{-1} \int\limits_{B_{min}^{(abs)}/B_0}^{B_{max}^{(abs)}/B_0} db' \sum_{j=1}^{j_{max}} \frac{\hat{H}_j^2}{\hat{I}_j}, \tag{4}$$

$$\hat{H}_j = \frac{1}{b'} \int\limits_{s_j^{(min)}}^{s_j^{(max)}} \frac{ds}{B} \sqrt{b' - \frac{B}{B_0}} \left(4\frac{B_0}{B} - \frac{1}{b'} \right) |\nabla\psi| k_G, \quad \hat{I}_j = \int\limits_{s_j^{(min)}}^{s_j^{(max)}} \frac{ds}{B} \sqrt{1 - \frac{B}{B_0 b'}}. \tag{5}$$

Here $k_G = (\mathbf{h} \times (\mathbf{h} \cdot \nabla)\mathbf{h}) \cdot \nabla\psi/|\nabla\psi|$ is the geodesic curvature of a magnetic field line with the unit vector $\mathbf{h} = \mathbf{B}/B$. The quantity ϵ_{eff} is calculated by integration over the magnetic field line length, s, over the sufficiently large interval $0 \div L_s$, and by integration over the perpendicular adiabatic invariant of trapped particles, $J_\perp$, by means of the variable b'. Here, $B_{min}^{(abs)}$ and $B_{max}^{(abs)}$ are the minimum and maximum values of B within the interval $0 \div L_s$. The quantities $s_j^{(min)}$ and $s_j^{(max)}$ within the sum over j in (5) correspond to the turning points of trapped particles. For a magnetic field originally available in real space coordinates, there is no necessity in a field transformation to magnetic coordinates. In this case, formulas (3) - (5) must be supplemented with the magnetic field line equations as well as with the equations for the vector $\mathbf{P} \equiv \nabla\psi$ (see Ref.[2])

$$\frac{dP_i}{ds} = -\frac{1}{B}\frac{\partial B^j}{\partial \xi^i}P_j \,, \qquad (6)$$

where B^j are the contravariant components of $\mathbf{B}$ in a real-space coordinates ξ^i, and $P_j = \partial\psi/\partial\xi^j$ are the covariant components of $\mathbf{P}$.

Normalized diffusion coefficients

An effective ripple can also be determined in a form without the formal magnetic surface radius given by (2). This can be done by scaling the $1/\nu$ flux with the use of some other independent diffusion flux. Transport fluxes at low collision frequencies are linked to the charged particle drift motion. Therefore, for the correct comparison of these fluxes, one has to use a normalization flux which takes into account the plasma geometry but at the same time is independent of the drift motion. The usual 'classical' diffusive flux for heat or particles across the magnetic field is the most appropriate one for such a normalization. For an arbitrary magnetic field the 'classical' particle diffusion flux has the following form [3]

$$F_\perp = F_\perp^{(conv)}/\xi, \qquad F_\perp^{(conv)} = \frac{nc^2}{\sigma_\perp B_0^2}\frac{dp}{d\psi}\langle|\nabla\psi|\rangle \,, \qquad (7)$$

with

$$\xi = \langle|\nabla\psi|\rangle^2\left(B_0^2\left\langle\left(\frac{\nabla\psi}{B}\right)^2\right\rangle\right)^{-1}, \quad \langle A \rangle = \lim_{L_s\to\infty}\left(\int_0^{L_s}\frac{ds}{B}\right)^{-1}\int_0^{L_s}\frac{ds}{B}A \,. \qquad (8)$$

Taking (7) and the $1/\nu$ particle flux in the form (1), one can obtain

$$\frac{F_n}{F_\perp} = \frac{\epsilon_{\mathrm{effn}}^{3/2}}{\epsilon_{\mathrm{h}}^{3/2}}\frac{F_n^{(conv)}}{F_\perp^{(conv)}}, \qquad \epsilon_{\mathrm{effn}}^{3/2} = \xi\epsilon_{\mathrm{eff}}^{3/2} = \left(B_0^2\left\langle\left(\frac{\nabla\psi}{B}\right)^2\right\rangle\right)^{-1}\hat{\epsilon}_{\mathrm{eff}}^{3/2} \,. \qquad (9)$$

The ratio $F_n^{(conv)}/F_\perp^{(conv)}$ is independent of the formal radius since the derivative $\langle|\nabla\psi|\rangle\frac{\partial}{\partial\psi}$ enters the numerator as well as the denominator when taking into account (2). Therefore, the effect of the magnetic field geometry on the $1/\nu$ transport coefficients manifests itself through the factor $\epsilon_{\mathrm{effn}}^{3/2}$ only. The ratio of $\epsilon_{\mathrm{effn}}^{3/2}$ over $\epsilon_{\mathrm{eff}}^{3/2}$ is given by the factor ξ which turns out to be rather close to unity. For example, for the magnetic configurations considered in [1] (original Helias, quasi-helically symmetric stellarator, torsatron U-3M) this value is in the limits $0.8\div1.0$. This is valid for both, B_0 defined as $\langle B \rangle$ or as $1/\langle 1/B\rangle$.

Presentation in Boozer coordinates

To consider transport processes in configurations with finite beta equilibria, the Boozer magnetic coordinates are often used. The contravariant and covariant representations of $\mathbf{B}$ are [4]

$$\mathbf{B} = \nabla\psi \times \nabla\theta - \iota\nabla\psi \times \nabla\varphi = F\nabla\varphi + I\nabla\theta + \beta_*\nabla\psi . \tag{10}$$

Here θ and φ are the Boozer angle-like magnetic coordinates, $cF(\psi)/2$ and $cI(\psi)/2$ are the poloidal (external with respect to the magnetic surface) and toroidal electric currents, ι is the rotational transform in units of 2π, and $\Phi = 2\pi\psi$ is the toroidal magnetic flux. The well known expressions for the metric tensor determinant and the field line elements are

$$\sqrt{g} = \frac{1}{\nabla\psi \times \nabla\theta \cdot \nabla\varphi} = \frac{F + \iota I}{B^2}, \qquad \frac{ds}{B} = \sqrt{g}d\varphi, \qquad \theta = \theta_0 + \iota\varphi . \tag{11}$$

With the use of (10), one gets the quantity $|\nabla\psi|k_G$ as

$$|\nabla\psi|k_G = \frac{1}{B^2}(\mathbf{B} \times \nabla B) \cdot \nabla\psi = \frac{1}{F + \iota I}\left(I\frac{\partial B}{\partial\varphi} - F\frac{\partial B}{\partial\theta}\right). \tag{12}$$

With the Boozer spectrum of B, with the quantities F and I, and with expressions (11) for $\sqrt{g}$ and (12) for $|\nabla\psi|k_G$, the parameter $\hat{\epsilon}_{\text{eff}}^{3/2}$ in (3) can be computed. So, the quantity $\hat{\epsilon}_{\text{eff}}^{3/2}$ represents that part of $\epsilon_{\text{eff}}^{3/2}$ for which the magnetic field confinement properties are determined through the Boozer spectrum of B and through F and I. It follows from (3) that for a full account of the effect of plasma geometry on $\epsilon_{\text{eff}}^{3/2}$ the quantity $|\nabla\psi|$ is also essential. For that purpose, one can directly use the Boozer spectrum for $|\nabla\psi|$. Alternatively, one can use the relation with the normal to the magnetic surface, $\mathbf{N}$,

$$\nabla\psi = \frac{1}{\sqrt{g}}\mathbf{N}, \quad \mathbf{N} = \frac{\partial\mathbf{r}}{\partial\theta} \times \frac{\partial\mathbf{r}}{\partial\varphi} \tag{13}$$

and make use of the Boozer spectra for the coordinates of a magnetic surface. At last, as it follows from (2) and from the equivalence of averages over the magnetic field line and over the volume between two neighboring magnetic surfaces, $\langle|\nabla\psi|\rangle$ is given as

$$\langle|\nabla\psi|\rangle = S\left(\frac{dV}{d\psi}\right)^{-1} = S\left(2\pi\frac{dV}{d\Phi}\right)^{-1}, \tag{14}$$

with S and V being the magnetic surface area and the volume inside this surface, respectively.

Mono-energetic diffusion coefficient

Another approach to compute an effective ripple is a Monte-Carlo computation of the $1/\nu$ neoclassical diffusion coefficients. These computations are often associated with the so called mono-energetic diffusion coefficient $D(E)$. According to the definition of $D(E)$ given by Boozer [5], $D_B(E)$, the average flux is given as

$$\Gamma_\iota = SF_n = -\int\frac{d\theta_0 d\varphi}{\mathbf{B} \cdot \nabla\varphi}\int_0^\infty D_B(E)\frac{\partial f_m}{\partial\psi}4\pi v^2 dv = -\frac{dV}{d\psi}\int_0^\infty D_B(E)\frac{\partial f_m}{\partial\psi}4\pi v^2 dv, \tag{15}$$

where $E = mv^2/2$ is the particle energy (see Eq.(14) in Ref. [5]). In analogy to this definition, one can define the mono-energetic diffusion coefficient in arbitrary coordinates as

$$F_n = -\int_0^\infty D(E,\psi)\frac{\partial f_m}{\partial r}4\pi v^2 dv \tag{16}$$

with the formal radius r defined in (2). Transforming the integration over $z = mv^2/(2T)$ in (1) back to the integration over v, we obtain

$$F_n = -\frac{\sqrt{2}}{9\pi} \frac{v_T^2 \rho_L^2}{\nu R^2} \epsilon_{\text{eff}}^{3/2} \int\limits_0^\infty \frac{1}{A(E/T)} \left(\frac{E}{T}\right)^2 \frac{\partial f_m}{\partial r} 4\pi v^2 dv \ . \tag{17}$$

A comparison with (16) gives for the $1/\nu$ regime

$$D(E, \psi) = \frac{\sqrt{2}}{9\pi} \frac{v^2 \rho^2 \epsilon_{\text{eff}}^{3/2}}{\nu A(E/T) R^2}, \tag{18}$$

where $v = \sqrt{2E/m}$ and $\rho = mcv/(eB_0)$. The pitch angle diffusion coefficient is linked to $\nu_\perp$ from the NRL Formulary according to $\nu A(E/T) = \nu_\perp/2$.

Peculiarity of Boozer diffusion coefficients

It is well known, that the Hamiltonian of the guiding center motion with constant magnetic moment shows that the corresponding particle orbits are completely determined by the structure of B. Therefore, in Boozer coordinates, the guiding center equations do not contain other metric elements besides $\sqrt{g}$. Evidently, the quantity $\nabla\psi$ does not enter these equations. However, when calculating the particle flux across a magnetic surface, the geometry of this surface must be taken into account. This leads to the appearance of $\nabla\psi$ in the results in Boozer coordinates. In the work of Boozer, the role of the magnetic surface geometry (function ψ) is eliminated from the Boozer diffusion coefficient and transferred to the total flux expression (15). From (15) and (16), taking into account (2) and (14) follows

$$D_B(E) = D(E, \psi)\langle|\nabla\psi|\rangle^2 \ . \tag{19}$$

So, the quantity $\epsilon_{\text{eff}}^{3/2}$ in (3) corresponds to $D(E, \psi)$, whereas the quantity $\hat{\epsilon}_{\text{eff}}^{3/2}$ corresponds to the Boozer diffusion coefficient $D_B(E)$.

Summary

There exists at least two forms for the presentation of $\epsilon_{\text{eff}}^{3/2}$: A form related to a formal radius of a magnetic surface and a form related to a normalization with respect to the 'classical' diffusion flux. Both forms give sufficiently close results and so both can be used for a practical evaluation of the effective ripple. It is also shown here that the Boozer mono-energetic diffusion coefficient differs by a factor of $\langle|\nabla\psi|\rangle^2$ from the mono-energetic diffusion coefficient related to the average flux density across the magnetic surface. For calculations in Boozer coordinates, the effective ripple turns out to depend on the spectra of B and $\nabla\psi$.

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ОЦІНКА ЕФЕКТИВНОЇ ГОФРОВКИ У СТЕЛАРАТОРАХ

В. В. Нємов, С. В. Касілов, В. Кернбіхлер, М. Ф. Хейн

У роботі проведено аналіз коефіцієнтів дифузії та обговорюються особливості різних варіантів ефективної гофровки поля, ε_{eff} .