ASYMPTOTIC DESCRIPTION OF PLASMA TURBULENCE: KRYLOV – BOHOLIUBOV METHODS AND QUASI-PARTICLES

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The asymptotic theory of charged particle motion in electromagnetic fields is developed for the general case of finite Larmor-radius effects by means of Krylov-Boholiubov averaging method. The correspondence between the general asymptotic methods, elaborated by M. Krylov and M. Boholiubov, the quasi-particle description and gyrokinetics is established. Such a comparison is used to shed more light on the physical sense of the reduced Poisson equation, introduced in gyrokinetics, and the particle polarization drift. It is shown that the modification of the Poisson equation in the asymptotic theory is due to the non-conservation of the magnetic moment and gyrophase tremblings. It is shown that the second-order modification of the adiabatic invariant can determine the conditions of global plasma stability and introduces new nonlinear terms into the reduced Poisson equation. Such a modification is important for several plasma orderings, e.g. MHD type ordering. The feasability of numerical simulation schemes in which the polarization drift is included into the quasi-particle equations of motion, and the Poisson equation remains unchanged is analyzed. A consistent asymptotic model is proposed in which the polarization drift is included into the quasi-particle equations of motion and the particle and quasi-particle velocities are equal. It is shown that in such models there are additional modifications of the reduced Poisson equation. The latter becomes even more complicated in contrast to earlier suggestions.

1 Introduction

Plasmas can be considered as a paradigm of a very complex dynamic system. Therefore simplified theoretical descriptions and numerical schemes based on asymptotic methods are of great importance. The asymptotic theory of charged particle motion in electromagnetic fields is developed for the general case of finite Larmor-radius effects by means of Krylov-Boholiubov averaging method. The correspondence between the general asymptotic methods, elaborated by M. Krylov and M. Boholiubov, the quasi-particle description and gyrokinetics is established. Such a comparison is used to shed more light on the physical sense of the reduced Poisson equation, introduced in gyrokinetics, and the particle polarization drift. It is shown that the modification of the Poisson equation in the asymptotic theory is due to the non-conservation of the magnetic moment and gyrophase tremblings. It is shown that the second-order modification of the adiabatic invariant can determine the conditions of global plasma stability and introduces new non-linear terms into the reduced Poisson equation. Such a modification is important for several plasma orderings, e.g. MHD type one. The feasibility of numerical simulation schemes in which the polarization drift is included into the quasi-particle equations of motion, and the Poisson equation remains unchanged is analysed. A consistent asymptotic model is proposed in which the polarization drift is included into the quasi-particle equations of motion and the particle and quasi-particle velocities are equal. It is shown that in such models there are additional modifications of the reduced Poisson equation. The latter becomes even more complicated in contrast to earlier suggestions.

2 Particles, Guiding Centres and Quasi-Particles

Let us consider the case of a non-relativistic particle of mass m and charge q in a constant magnetic field $\vec{B} \equiv B\vec{b}$ and in the potential electric field $\vec{E} = -\vec{\nabla}\Phi$, described by a scalar potential Φ. The Newton-Lorentz equations of motion are

$$\dot{\vec{r}} = \vec{v} \; , \quad \dot{\vec{v}} = \Omega\vec{v} \times \vec{b} \, + \, \vec{a} \; , \tag{1}$$

with $\Omega = qB/mc$ the Larmor frequency, and $\vec{a} = q\vec{E}/m = -\vec{\nabla}\theta$ the particle acceleration in the electric field.

The magnetic field is assumed to be strong in the following sense. The acceleration $\vec{a}$ is treated as a small perturbation of particle motion, $a \sim \lambda v_\perp \Omega$, $\lambda << 1$, with slow time dependence, $\partial_t \vec{a} \sim \lambda\Omega\vec{a}$, where $\vec{v} = \vec{v}_\perp + u\vec{b}$, $u = \vec{b}\cdot\vec{v}$, and the index $\perp$ ($||$) is for vector components across (along) $\vec{B}$ and their magnitudes. The electric field is assumed to vary weakly in space along the magnetic field direction at scales of Larmor radius order, $r_L\vec{b} \cdot \vec{\nabla}\vec{a} \sim \lambda\vec{a}$, with $r_L^2 = v_\perp^2/\Omega^2$. However no assumptions are made about the electric field variation across the magnetic field, which is beyond the framework of classical references.

The cylindrical velocity coordinates $v_\perp, \alpha, u$, $\vec{v}_\perp = v_\perp(cos\alpha\ \hat{x} + sin\alpha\ \hat{y})$, (with the Cartesian reference system of orthogonal and unitary vectors $\hat{x}$, $\hat{y}$ and $\vec{b}$), together with the guiding-centre space coordinates $\vec{R} = (X, Y, Z)$, $\vec{R} = \vec{r} - \vec{r}_L$, with $\vec{r}_L = \vec{b} \times \vec{v}/\Omega$ the Larmor radius-vector, are convenient as phase space coordinates. In such coordinates, the equations of particle motion take the standard form of a nonlinear dynamical system with rapidly varying phase α:

$$\begin{aligned} \dot{\vec{R}} \; &= \; u\vec{b} \, + \, \vec{b} \times \vec{\nabla}\theta/\Omega \; , \quad \dot{u} \; = \; -\nabla_{||}\theta \; , \\ \dot{\epsilon}_\perp \; &= \; \Omega\partial_\alpha\theta \; , \quad \dot{\alpha} \; = \; -\Omega(1 \, + \, \partial_{\epsilon_\perp}\theta) \; , \end{aligned} \tag{2}$$

where $\epsilon_\perp \equiv v_\perp^2/2$ is the transverse kinetic energy. These equations govern the motion of the guiding centre. They can be also written compactly as

$$\dot{\vec{x}} = \vec{X}(\vec{x}, \alpha, t) \; , \quad \dot{\alpha} = -\Omega - \nu(\vec{x}, \alpha, t) \; , \tag{3}$$

with $\vec{x} = (\vec{R}, u, \epsilon_\perp)$, and the notations $\vec{X}$ and ν being evident. The small parameter λ can be associated with the perturbing potential θ.

3 Basic Equations of Averaged Motion

The change of variables from $(\vec{x}, \alpha)$ to $(\bar{\vec{x}}, \bar{\alpha})$,

$$\vec{x} = \bar{\vec{x}} + \delta\vec{x}(\bar{\vec{x}}, \bar{\alpha}, t) \; , \quad \alpha = \bar{\alpha} + \delta\alpha(\bar{\vec{x}}, \bar{\alpha}, t) \; , \tag{4}$$

is required to get new equations of motion

$$\dot{\bar{\vec{x}}} = \bar{\vec{X}}(\bar{\vec{x}}, t) \; , \quad \dot{\bar{\alpha}} = -\Omega - \bar{\nu}(\bar{\vec{x}}, t) \; , \tag{5}$$

in which the dependence on the rapidly varying phase $\bar{\alpha}$ is removed from the right-hand sides (M. Krylov and M. Boholubov). The above change of variables can be called averaging transformation in this context, while the new equations of motion can be called averaged equations or equations of averaged motion. Such an interpretation suggests the following natural normalization conditions:

$$\int_0^{2\pi} d\alpha\, \delta\vec{x}(\bar{\vec{x}},\alpha,t) \;=\; 0\;, \quad \int_0^{2\pi} d\alpha\, \delta\alpha(\bar{\vec{x}},\alpha,t) \;=\; 0\;, \tag{6}$$

when the new variables contain all the averaged motion. Then the mean asymptotic value $\bar{Q}$ of any function $Q(\vec{x},\alpha,t)$, $\bar{\vec{X}}$ and $\bar{\nu}$ in particular, is given by the exact expression

$$\bar{Q} = \frac{1}{2\pi}\int_0^{2\pi} d\bar{\alpha} Q(\bar{\vec{x}} + \delta\vec{x}(\bar{\vec{x}},\bar{\alpha},t),\; \bar{\alpha} + \delta\alpha(\bar{\vec{x}},\bar{\alpha},t),\; t)\;, \tag{7}$$

while the tremblings satisfy the following exact equations:

$$[\,\partial_t + \bar{\vec{X}}\cdot\partial_{\bar{\vec{x}}} - (\Omega+\bar{\nu})\,\partial_{\bar{\alpha}}\,]\,\delta\vec{x} \;=\; \vec{X} - \bar{\vec{X}}\;,$$

$$[\,\partial_t + \bar{\vec{X}}\cdot\partial_{\bar{\vec{x}}} - (\Omega+\bar{\nu})\,\partial_{\bar{\alpha}}\,]\,\delta\alpha \;=\; \nu - \bar{\nu}\;. \tag{8}$$

According to the general asymptotic approach developed by Boholiubov and Krylov, the tremblings $\delta\vec{x}$ and $\delta\alpha$ and any mean values can be found as asymptotic series in increasing powers of a small positive parameter λ associated with $\vec{X}$, ν and their explicit slow time dependence.

In the first order:

$$\delta\vec{x} = -\frac{1}{\Omega}\int d\bar{\alpha}\;[\vec{X}(\bar{\vec{x}},\bar{\alpha},t) - \vec{X}_0]\;,\quad \delta\alpha = -\frac{1}{\Omega}\int d\bar{\alpha}\;[\nu(\bar{\vec{x}},\bar{\alpha},t) - \nu_0]\;, \tag{9}$$

$$\bar{Q} = Q_0 \;+\; \{(\delta\vec{x}\cdot\partial_{\bar{\vec{x}}} + \delta\alpha\cdot\partial_{\bar{\alpha}})Q\}_0 \;= Q_0 \;+\; \{(\delta\vec{x}\cdot\partial_{\bar{\vec{x}}} + \frac{\nu-\nu_0}{\Omega})Q\}_0\,. \tag{10}$$

The following notation is used for any $Q(\vec{x},t)$:

$$Q_n \equiv \{Q\}_n \equiv \frac{1}{2\pi}\int d\bar{\alpha} e^{in\bar{\alpha}} Q(\bar{\vec{x}},\bar{\alpha},t)\;. \tag{11}$$

If the tremblings are incompressible (which is true for the particular case studied here)

$$\partial_{\bar{\vec{x}}}\cdot\delta\vec{x} + \partial_{\bar{\alpha}}\cdot\delta\alpha = 0 \quad\rightarrow\quad \bar{Q} = Q_0 \;+\; \partial_{\bar{\vec{x}}}\cdot\{\delta\vec{x}Q\}_0\,. \tag{12}$$

The latter expression for the mean values is convenient since the trembling $\delta\alpha$ does not enter it explicitly.

Thus

$$\delta I = -(\theta-\theta_0)/\Omega\;,\quad \delta\alpha = -\partial_{\bar{I}}\delta\hat{I}\;,\quad \delta\hat{I} \equiv \int d\bar{\alpha}\delta I\;,$$

$$\delta u = -\partial_{\bar{Z}}\delta\hat{I}\;,\quad \delta Z = -\frac{1}{\Omega}\int d\bar{\alpha}\delta u\;,$$

$$\delta\vec{R}_\perp = -\frac{1}{\Omega}\,\vec{b}\times\vec{\nabla}\delta\hat{I}\ ,$$

while the averaged equations are

$$\dot{\bar{\vec{R}}} = \bar{u}\vec{b} + \vec{b}\times\vec{\nabla}\theta_s/\Omega\ ,\quad \dot{\bar{u}} = -\nabla_\parallel\theta_s\ ,$$

$$\dot{\bar{\epsilon}}_\perp = 0\ ,\quad \dot{\bar{\alpha}} = -\Omega(1 + \partial_{\bar{\epsilon}_\perp}\theta_s)\ . \tag{13}$$

In the lowest zeroth order there is no need to distinguish between a particle quantity and its mean asymptotic value. Therefore if the latter is conserved then both are invariant. In particular $I = \bar{I} = const$ in the lowest order. This means that in the non-relativistic limit the transverse kinetic energy $m\epsilon_\perp$ or the magnetic moment $m\epsilon_\perp/B$ is the lowest order adiabatic invariant. In the relativistic theory this will be $m^2\epsilon_\perp/B$, the quantity proportional to the magnetic field flux across the Larmor circle.

The tremblings of particle quantities about their mean values are taken into account in the first order. Thus the particle magnetic moment is not invariant in this order even though its mean value appears to be still conserved, $\bar{I} = const$. If $\bar{I}$ is expressed in terms of original particle (or guiding-centre) variables, the first-order particle adiabatic invariant follows:

$$\bar{I} = I - \delta I = I + \frac{1}{\Omega}[\theta(\vec{r},t) - \frac{1}{2\pi}\int_0^{2\pi} d\alpha'\theta(\vec{r} + \frac{1}{\Omega}\vec{b}\times(\vec{v}-\vec{v}'),t)]\ , \tag{14}$$

where $\vec{v}_\perp = v_\perp(cos\alpha, sin\alpha)$, $\vec{v}'_\perp = v_\perp(cos\alpha', sin\alpha')$.

Here θ_s is the potential of the mean acceleration,

$$\theta_s = \theta_0 - \frac{1}{2}\,\Omega\partial_{\bar{I}}\{(\delta I)^2\}_0 - \frac{1}{2}\,\vec{\nabla}\cdot\{\delta I\vec{b}\times\vec{\nabla}\delta\hat{I}\}_0\ , \tag{15}$$

$$\bar{\vec{a}} = -\vec{\nabla}\theta_s\ ,$$

while the mean potential is

$$\bar{\theta} = \theta_0 - \Omega\partial_{\bar{I}}\{(\delta I)^2\}_0 - \vec{\nabla}\cdot\{\delta I\vec{b}\times\vec{\nabla}\delta\hat{I}\}_0\ , \tag{16}$$

or $\theta_s = (\theta_0 + \bar{\theta})/2$. The potential of the mean acceleration and the mean potential are equal only in the lowest zeroth order, $\theta_s = \bar{\theta} = \theta_0$. In the first order they differ.

The averaged equations are in agreement with the quasi-particle description [4]. But unlike the latter they provide information about the time evolution of $\bar{\alpha}$.

The mean particle velocity

$$\{\Omega\partial_\alpha P\}_m = \{(\partial_t + \vec{X}\cdot\partial_{\vec{x}} - \nu\partial_\alpha)P\}_m - \dot{\bar{P}}\ , \tag{17}$$

where $\{Q\}_m$ means the asymptotic averaging as well as $\bar{Q}$, for $P = \vec{v}_\perp$ and $P = \partial_\alpha\vec{v}_\perp$:

$$\bar{\vec{v}} = \bar{u}\vec{b} - \vec{\nabla}\theta_s\times\vec{b}/\Omega - \frac{d}{dt}\vec{\nabla}_\perp\theta_0/\Omega^2\ . \tag{18}$$

agrees with the quasi-particle description. The first term $\bar{u}\vec{b}$ on the right describes particle streaming along $\vec{B}$, the second term is the electric drift velocity modified owing to FLR (finite-Larmor-radius) effects (θ_s instead of θ), the third term is the polarization drift velocity also modified owing to FLR effects (θ_0 instead of θ). The polarization drift effect and the first-order renormalization of the electric drift velocity (θ_s instead of θ_0) are of the same order. One can take

$$\frac{d}{dt} = \partial_t + \bar{u}\vec{b}\cdot\vec{\nabla} - \frac{1}{\Omega}\,\vec{\nabla}\theta_0 \times \vec{b}\cdot\vec{\nabla}$$

in the polarization drift term within the accuracy of interest.

If one considers the guiding-centre velocity $\vec{V} = \dot{\vec{R}}$, its mean value

$$\bar{\vec{V}} = \bar{u}\vec{b} - \vec{\nabla}\theta_s \times \vec{b}/\Omega \tag{19}$$

is not equal to the mean particle velocity. The equation for the guiding-centre velocity does not contain terms which might be associated with a particle polarization drift. However, one is accustomed to encounter polarization drift in the guiding-centre theory.

One can imagine the following computation scheme. From given initial conditions for particles the initial conditions for guiding centres are found, as well as the initial charge density which determines the initial electric field via the Poisson equation. Then the averaging transformation is used to find the initial conditions for the averaged motion. From the latter the mean coordinates and velocities are pushed one step forward in time by means of averaged equations of motion taking account of initial electric field.

From the new mean coordinates in the phase space new guiding-centre coordinates are found by means of averaging transformation, and then new particle coordinates by means of guiding-centre transformation. These updated particle coordinates are used to compute the new values of the charge density and the electric field.

Then the mean coordinates are pushed one step forward in time again, and so on. The numerical computation loop is closed.

For realistic applications this computation scheme is to be modified yet by taking advantages of the quasi-particle description. One assumes that rapidly varying in time electric fields can be disregarded at all, and that the self-consistent interaction between the particles via the low-frequency (slowly evolving in time) electric fields is significant only. Then the time evolution of the mean phase can be excluded from the consideration, and the electric field can be calculated self-consistently from the quasi-particle density by means of Poisson equation.

A microscopic quasi-particle density in the reduced phase space $\vec{x}$ is introduced as the mean value of the microscopic guiding-centre density F_c integrated over the phase:

$$G(\vec{x},t) \equiv \int_0^{2\pi} d\alpha \bar{F}_c(\vec{x},\alpha,t)\ . \tag{20}$$

It can be interpreted as the reduced averaged probability density for guiding centres. One can relate the mean guiding-centre density

$$\bar{F}_c(\vec{x},\alpha,t) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\bar{\alpha} F_c(\vec{x},\alpha,t) \tag{21}$$

to the quasi-particle density G. In the first order

$$\bar{F}_c = \frac{1}{2\pi}\left[\, G - \partial_{\vec{x}}\cdot(\delta\vec{x}G) - \partial_\alpha(\delta\alpha G)\,\right], \tag{22}$$

which is the relationship between guiding centres an quasi-particles, or in more detail

$$\bar{F}_c = \frac{1}{2\pi}\left[\, 1 - \delta\vec{R}\cdot\vec{\nabla} - \delta u\partial_u - \delta\epsilon_\perp\partial_{\epsilon_\perp}\,\right] G\,.$$

The mean particle density follows

$$\bar{F}_p(\vec{r},\vec{v},t) = \bar{F}_c(\vec{r}-\vec{b}\times\vec{v}/\Omega, u, \epsilon_\perp, \alpha, t)\,, \tag{23}$$

which is the relationship between particles and guiding centres.

The reduced Poisson equation

$$\vec{\nabla}\cdot\vec{E} = 4\pi\Big[\sum q\int d\vec{v}\,\bar{F}_p(\vec{r},\vec{v},t) + \rho^{ext}\Big] \tag{24}$$

takes into account the tremblings of guiding-centre coordinates, $\delta\alpha$ and $\delta\epsilon_\perp$ in particular. Such tremblings produce the difference between the mean guiding-centre density and the quasi-particle density. The tremblings $\delta\vec{R}$ and δu do not contribute to the averaged Poisson equation for the particular electrostatic case considered. Thus the modification of the Poisson equation arises from the velocity phase tremblings and the first-order modification of particle adiabatic invariant (the transverse kinetic energy or the magnetic momentum). Therefore, it is not necessarily related to the polarization particle drift.

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АСИМПТОТИЧНИЙ ОПИС ПЛАЗМОВОЇ ТУРБУЛЕНТНОСТІ: МЕТОДИ КРИЛОВА – БОГОЛЮБОВА ТА КВАЗІ-ЧАСТИНКИ

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Розвинено асимптотичну теорію руху заряджених частинок у електромагнітних полях за допомогою методу усереднення Крилова – Боголюбова, яка є справедливою для частинок із довільним Ларморівським радіусом. Установлено відповідність між загальними асимптотичними методами, розробленими М. Криловим та М. Боголюбовим, та квазі-частинковим описом і гірокінетикою. Таке порівняння використано для висвітлення фізичного сенсу зведеного рівняння Пуасона, що вводиться в гірокінетиці, та поляризаційного дрейфу частинок. Показано, що модифікація рівняння Пуасона в асимптотичній теорії зумовлена незбереженням магнітного моменту та гірофазового коливання. Показано, що модифікація другого порядку в адіабатичному інваріанті може визначати умови глобальної стійкості плазми і давати нові нелінійні члени у зведеному рівнянні Пуасона. Проаналізовано здійсненність схем чисельного моделювання, в яких поляризаційний дрейф включено до рівнянь руху квазі-частинок, а рівняння Пуасона залишається незмінним. Пропонується послідовна асимптотична модель, у якій поляризаційний дрейф включено до рівнянь руху квазі-частинок, а швидкість частинки та квазі-частинки є однаковою. Показано, що в таких моделях виникають додаткові модифікації зведеного рівняння Пуасона. Всупереч попереднім припущенням, останнє стає навіть більш складним.