REDUCED MHD EQUATIONS FOR ALFVÉN EIGENMODES IN STELLARATORS

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Reduced magnetohydrodynamic (MHD) equations are derived, in which the plasma compressibility is taken into account, but fast magnetoacoustic waves are excluded. For the sake of simplicity all terms associated with the pressure gradient and the plasma current are disregarded. However, the continuous spectrum of the obtained equations is shown to coincide exactly with the continuous spectrum of the full MHD equations. First results of the code COBRAS (COntinuum BRanches of Alfvén and Sound waves) intended for calculation of coupled Alfvén and slow continua, are presented. Effect of the compressibility on the Alfvén continuum in Wendelstein-line stellarators is studied.

1. INTRODUCTION

The plasma compressibility is known to affect the Alfvén spectrum in toroidal plasmas, leading to coupling of Alfvén waves with slow magnetoacoustic waves when the geodesic curvature of magnetic field lines is non-zero [1]. This effect was studied in a number of works [2–6]. In particular, it was shown that when β is high(β is the ratio of the plasma pressure to the magnetic field pressure), the compressibility affects the continuum gap where toroidicity-induced Alfvén eigenmodes (TAE) reside, shifting it up in frequency [2,3]. In addition, compressibility opens new gaps at the points where branches of the Alfvén continuum intersect branches of the slow continuum, with specific eigenmodes residing in such gaps [4,5].

In this work, reduced magnetohydrodynamic (MHD) equations suitable for description of Alfvén eigenmodes coupled with slow waves are derived. We pursue the aim to obtain as simple equations as possible, which was achieved by eliminating fast magnetoacoustic waves, keeping pressure terms but disregarding all terms associated with the pressure gradient (which are responsible for interchange and ballooning modes rather than for slow waves). Properties of the MHD continuum in non-axisymmetric configurations are studied both analytically and with the new code COBRAS (COntinuum BRanches of Alfvén and Sound waves), which is an extension of the code COBRA [7]. The effect of compressibility on the Alfvén continuum in Wendelstein 7-AS is determined.

The structure of the work is as follows. The derivation of the reduced MHD equations is presented in Sec. 2. In Sec. 3, the continuous spectrum of the equations is studied, and the effect of the compressibility on the Alfvén continuum is revealed. Finally, in Sec. 4 the obtained results are summed up and discussed.

2. DERIVATION OF REDUCED MHD EQUATIONS

We proceed from the linearized ideal MHD equations in the form

$$-\rho\omega^2\vec{\xi} + \nabla\tilde{p} = \frac{1}{c}\vec{\tilde{j}}\times\vec{B}_0 + \frac{1}{c}\vec{j}_0\times\vec{\tilde{B}}, \tag{1}$$

$$\vec{\tilde{B}} = \nabla\times(\vec{\xi}\times\vec{B}_0), \tag{2}$$

$$\tilde{p} = -\vec{\xi}\cdot\nabla p_0 - \gamma p_0\nabla\cdot\vec{\xi}, \tag{3}$$

where tildes and subscripts "0" refer to perturbed and equilibrium quantities, respectively; the time dependence of the perturbed quantities is taken as $\exp(-i\omega t)$; $\vec{\xi}$ is the plasma displacement; $\vec{B}$ is the magnetic field; $\vec{j} = c/(4\pi)\nabla\times\vec{B}$ is the current density; p is the plasma pressure; ρ is the equilibrium mass density; γ is the ratio of specific heats (we assume all processes to be adiabatic, i.e., $p/\rho^\gamma = \text{const}$).

To eliminate the fast magnetoacoustic waves, we assume that $\omega \ll k_\perp v_A$, $k_\parallel \ll k_\perp$, and $\mathcal{K} \ll k_\perp$ (here v_A is the Alfvén velocity, $\vec{\mathcal{K}} = (\vec{b}\cdot\nabla)\vec{b}$ is the field line curvature, $\vec{b} = \vec{B}_0/B_0$, $k_\parallel$ and $k_\perp$ are the longitudinal and perpendicular wave numbers, respectively) and write Eq. (1) as follows:

$$-\rho\omega^2\vec{\xi} + \nabla\tilde{p} = -\frac{1}{4\pi}\left[\nabla\left(\vec{B}_0\cdot\vec{\tilde{B}}\right) - \left(\vec{\tilde{B}}\cdot\nabla\right)\vec{B}_0 - \left(\vec{B}_0\cdot\nabla\right)\vec{\tilde{B}}\right]. \tag{4}$$

Given the adopted assumptions, the first term of the left-hand side and the last two terms of the right-hand side of the equation are small in comparison with the rest and can be neglected [8]. We obtain the equation of the balance of the perturbed pressure, $\nabla[\tilde{p} + 1/(4\pi)B_0\tilde{B}_\parallel] = 0$, from which we infer:

$$\tilde{p} + \frac{1}{4\pi}B_0\tilde{B}_\parallel = 0, \tag{5}$$

where $\tilde{B}_\parallel = \vec{b}\cdot\vec{\tilde{B}}$. Equation (5) states that the total pressure of the plasma and the magnetic field are not perturbed and, thus, eliminates the fast magnetoacoustic waves.

Taking the longitudinal component of Eq. (1) and using Eqs. (2) and (3), we obtain a relation between $\xi_\parallel$ and $\nabla\cdot\vec{\xi}$:

$$\xi_\parallel = -\frac{c_S^2}{\omega^2}\nabla_\parallel(\nabla\cdot\vec{\xi}), \tag{6}$$

where $\nabla_\parallel = \left(\vec{b}\cdot\nabla\right)$, $c_S = (\gamma p_0/\rho)^{1/2}$ is the sound velocity. Using Eq. (2) to exclude $\tilde{B}_\parallel$, Eq. (3) to exclude $\tilde{p}$, and expressing $\vec{\xi}$ in terms of $\vec{\xi}_\perp$ and $\nabla\cdot\vec{\xi}$ by means of the equation $\vec{\xi} = \vec{\xi}_\parallel + \vec{\xi}_\perp$ and Eq. (6), we can write Eq. (5) in the form:

$$\left[1 + \frac{c_S^2}{v_A^2} + \frac{c_S^2}{\omega^2}B_0\nabla_\parallel\left(\frac{1}{B_0}\nabla_\parallel\right)\right]\nabla\cdot\vec{\xi} + 2\vec{\mathcal{K}}\cdot\vec{\xi}_\perp = 0, \tag{7}$$

Keeping only the first term of Eq. (7) and assuming the cylindrical geometry, we recover the well-known dispersion relation of the slow magnetoacoustic waves ($\omega = |k_\parallel|v_S$, where $v_S = c_S v_A/(v_A^2 + c_S^2)^{1/2}$ is the velocity of slow magnetoacoustic waves). The second term, as we will see, describes coupling with Alfvén waves.

To derive the second equation, we follow our previous work [7] and use the equation $\nabla\cdot\vec{j} = 0$, which we write in the form

$$\nabla\cdot\vec{\tilde{j}}_\perp + \vec{B}_0\cdot\nabla\frac{\tilde{j}_\parallel}{B_0} = 0. \tag{8}$$

We find $\tilde{j}_\parallel$ from the equations $\vec{\tilde{j}} = (c/4\pi)\nabla\times(\nabla\times\vec{A})$, where $\vec{A}$ is the vector potential of the wave. To find $\vec{\tilde{j}}_\perp$, we cross Eq. (1) with $\vec{b}$. Then we use the longitudinal and perpendicular components of the equation

$$\nabla\Phi - \frac{i\omega}{c}\vec{A} + \frac{i\omega}{c}\vec{\xi}\times\vec{B}_0 = 0, \tag{9}$$

(where Φ is the scalar potential of the wave), which follows from the assumption that the plasma conductivity is ideal, to express $A_\parallel$ and $\xi_\perp$, respectively. Then we make some simplifications. First, as $\beta \equiv 8\pi p_0/B_0^2 \ll 1$ in a Helias, we omit the terms including

∇p_0, retaining only those pressure terms which involve maximum-order radial derivatives (i.e., the terms that affect the continuous spectrum). Second, we omit all terms including $\vec{j}_0$ because we are interested in configurations with vanishing toroidal current, where the equilibrium current results only from ∇p_0. Third, we neglect $\vec{A}_\perp$, which can also be shown to be proportional to the plasma pressure and does not affect the continuous spectrum. The resulting system consists of two equations and has the form:

$$\omega^2\nabla\cdot\left(\frac{1}{v_A^2}\nabla_\perp\Phi\right)+B_0\nabla_\parallel\left\{\frac{1}{B_0^2}\nabla\cdot\left[B_0^2\nabla_\perp\left(\frac{1}{B_0}\nabla_\parallel\Phi\right)\right]\right\}$$
$$-\frac{2i\omega}{c}\nabla\cdot\left[\frac{c_s^2}{v_A^2}\left(\vec{\mathcal{K}}\times\vec{B}_0\right)\left(\nabla\cdot\vec{\xi}\right)\right]=0, \tag{10}$$

$$\left[1+\frac{c_s^2}{v_A^2}+\frac{c_s^2}{\omega^2}B_0\nabla_\parallel\left(\frac{1}{B_0}\nabla_\parallel\right)\right]\left(\nabla\cdot\vec{\xi}\right)+\frac{2ic}{\omega B_0^2}\left(\vec{\mathcal{K}}\times\vec{B}_0\right)\cdot\nabla\Phi=0, \tag{11}$$

Eq. (10) being obtained from the equation $\nabla\cdot\vec{\tilde{j}}=0$, whereas Eq. (11) follows from Eq. (7). When the curvature is negligible, Eqs. (10) and (11) describe Alfvén and sound waves, respectively. The last terms of the equations are responsible for the coupling of the Alfvén and sound waves through the curvature of the magnetic field lines.

3. ALFVÉN AND SOUND CONTINUA

Following the approach of Refs. [9,10], one can show that the continuous spectrum of the system given by Eqs. (10) and (11) consists of the eigenvalues of the following system of equations:

$$\left[\frac{\omega^2|\nabla\psi|^2}{v_A^2}+B_0\nabla_\parallel\left\{\frac{|\nabla\psi|^2}{B_0}\nabla_\parallel\right\}\right]\eta+\frac{c_S^2}{v_A^2}\mathcal{K}_s B_0^2\zeta=0, \tag{12}$$

$$\left[1+\frac{c_S^2}{v_A^2}+\frac{c_S^2}{\omega^2}\left(\vec{B}_0\cdot\nabla\right)\left(\frac{1}{B_0}\nabla_\parallel\right)\right]\zeta+\mathcal{K}_s\eta=0, \tag{13}$$

where $\eta=(ic/\omega)\partial\Phi/\partial\psi$ and $\zeta=\nabla\cdot\vec{\xi}$ can be considered as wave functions of Alfvén and sound waves, respectively; ψ is the toridal magnetic flux; $\mathcal{K}_s=(2/B_0^2)(\vec{\mathcal{K}}\times\vec{B}_0)\cdot\nabla\psi$ characterizes the geodesic curvature of the magnetic field lines. The equations include only angular derivatives of η and ζ; their eigenvalues depend on ψ considered as a parameter and produce branches of the continuum as ψ is varied. One can see that Eqs. (12) and (13) exactly coincide with the equations of the continuous spectrum of the full MHD [10]. Thus, the elimination of the fast magnetoacoustic waves has been done correctly in the sense that it has not distorted, at least, the continuous spectrum of the waves of interest.

In the cylindrical geometry the continuous spectrum consists of Alfvén branches, $\omega=\omega_{Am,n}(\psi)$, and sound branches, $\omega=\omega_{Sm,n}(\psi)$, where $\omega_{Am,n}=|k_{\|m,n}|v_A$; $\omega_{sm,n}=|k_{\|m,n}|v_S$; $k_{\|m,n}=(m\iota-n)/R_0$; m and n are the poloidal and toroidal mode numbers, respectively; $\iota=\iota(\psi)$ is the rotational transform [1]. As the mode numbers (m,n) are arbitrary, the continuum covers the total frequency range. The most important consequence of the mode coupling due to deviations from the cylindrical symmetry is the appearance of gaps in the continuum; such gaps can contain eigenfrequencies of the discrete spectrum.

To consider the appearance of the gaps, it is convenient to write Eqs. (12) and (13) in the matrix form. With this aim, we make the following Fourier expansion:

$$X=\sum_{\mu,\nu=-\infty}^{\infty}X_{\mu,\nu}\exp[i(m+\mu)\theta-i(n+\nu N)\phi], \tag{14}$$

$$h_{g,B} = 1 + \frac{1}{2} \sum_{\mu,\nu=-\infty}^{\infty} \epsilon_{g,B}^{(\mu\nu)} \exp(i\mu\theta - i\nu N\phi) \tag{15}$$

where N is the number of the field periods, X denotes either η or ζ, $h_g = g^{\psi\psi}/\bar{g}^{\psi\psi}$, $h_B = B/\bar{B}$, $g^{\psi\psi} = |\nabla\psi|^2$, and the overbar denotes the flux-surface average. Thus, we arrive at an eigenproblem of the form $\Lambda(\omega)\vec{z} = 0$, where $\vec{z}$ is an infinite-dimensional vector containing the Fourier harmonics of the dependent variables η and ζ, $\Lambda = \Lambda(\omega)$ is an infinite matrix (we do not present it here explicitly for the lack of space). The non-diagonal elements of Λ involve $\epsilon_B^{(\mu\nu)}$ or $\epsilon_g^{(\mu\nu)}$, vanish in the cylindrical geometry (in this case the spectrum of Λ consists of Alfvén and sound branches produced by diagonal elements) and are responsible for coupling of the modes in non-cylindrical geometry.

The presence of the coupling parameters leads to so-called "avoid-crossing phenomenon" [11], i.e. prevents crossing of the branches which intersect in the absence of the coupling. In numerical calculations with MHD codes, gaps resulting from avoided crossings of continuum branches of various nature (an Alfvén branch with another Alfvén one, an Alfvén branch with a sound one, or a sound branch with another sound one) were observed [2,4–6]. (We will refer to such gaps as Alfvén ones, Alfvén-sound ones, and sound ones, respectively.) As shown in Refs. [7,12], Alfvén gaps produced by avoided crossings of the continuum branches $\omega = \omega_{Am,n}(\psi)$ and $\omega = \omega_{Am+\mu,n+\nu}(\psi)$, where m and n are arbitrary, and μ and ν are fixed and determine the type of coupling, have a common envelope – an "absolute" gap free of any Alfvén branches. When the mode coupling is weak, the absolute (μ, ν) Alfvén gap is located in the vicinity of the line $\omega = \omega_A^{(\mu\nu)}(\psi)$, where $\omega_A^{(\mu\nu)} = \bar{v}_A|k^{(\mu\nu)}|$, $k^{(\mu\nu)} = (\mu\iota - \nu N)/(2R_0)$. Similar considerations show that Alfvén-sound and sound gaps also form similar radially extended structures – absolute Alfvén-sound and sound gaps. Indeed, one can show that those intersections of the continuum branches $\omega = \omega_{Am,n}(\psi)$ and $\omega = \omega_{Sm\pm\mu,n\pm\nu}(\psi)$ which satisfy the condition $k_{\|m,n}k_{\|m\pm\mu,n\pm\nu} < 0$ intersect for any m and n at the line $\omega = \omega_{AS}^{(\mu\nu)}(\psi)$, where $\omega_{AS}^{(\mu\nu)} = 2|k^{(\mu\nu)}|\bar{v}_S\bar{v}_A/(\bar{v}_A+\bar{v}_S)$. When $\epsilon_B^{(\mu\nu)}$ is non-zero, the non-diagonal elements of Λ couple the mentioned pairs of branches, producing gaps near this line (note that no frequency gaps appear at the crossings where $k_{\|m,n}k_{\|m\pm\mu,n\pm\nu} > 0$). The joint envelope of such gaps is an absolute Alfvén-sound gap corresponding to the coupling numbers (μ, ν). Arguments similar to that presented in Ref. [7] show that no continuum branch, either Alfvén or sound, can cross this gap. Considering the intersections of pairs of the sound branches $\omega = \omega_{Sm,n}(\psi)$ and $\omega = \omega_{Sm+\mu,n+\nu}(\psi)$ with arbitrary m and n, one can conclude that a similar structure – an absolute sound gap – appears near the line $\omega = \omega_S^{(\mu\nu)}(\psi)$, where $\omega_S^{(\mu\nu)} = \bar{v}_S|k^{(\mu\nu)}|$. Summing up the above, we conclude that the MHD continuum is cut by voids (absolute gaps), which can be divided into three types: Alfvén gaps, where there are no Alfvén branches, sound gaps, where there are no sound branches, and Alfvén-sound gaps, where there are neither Alfvén nor sound branches.

To obtain realistic patterns of the MHD continuum in stellarator configurations, the code COBRAS was developed. The code extends the previously developed code COBRA [7] by involving the plasma compressibility and is intended for solving Eqs. (12) and (13). Like COBRA, the code COBRAS takes advantage of the fact that m and n enter the matrix Λ only through $k_\|$. This enables the code to delimit the continuum very accurately by solving the eigenproblem $\Lambda(\omega; \psi, k_\|)\vec{z} = 0$ truncated to a finite dimension for different values $k_\|$ and ψ and selecting the eigenvalues that are distorted by the truncation to the least extent (see also the detailed description of the COBRA code in Ref. [7]). To interpret the results, we have to divide the continuum into the Alfvén and sound parts. When the mode coupling is strong, this division is, to some extent, a matter of convention, especially, near Alfvén-sound gaps, where continuous transitions of Alfvén branches to sound ones and vice versa take place. We consider the branches with one of the $\eta_{m,n}$ components dominating as Alfvén ones and the branches with one of the $\zeta_{m,n}$ components dominating as sound ones (perhaps, it would be more exact to refer to them as Alfvén-dominated and sound-dominated branches).

Results of calculations of the MHD continuum in Wendelstein 7-AS, shot #43348, with COBRAS are shown in Figs. 1, 2. We observe that the high-frequency part of the continuum is weakly affected by the compressibility. But the low-frequency part of the spectrum is strongly affected. First, sound gaps and Alfven-sound gaps appear. Second, the Alfvén gaps in this region are shifted up and narrowed. In particular, the (1,0) Alfvén gap is shifted up by 12% and narrowed by 35%.

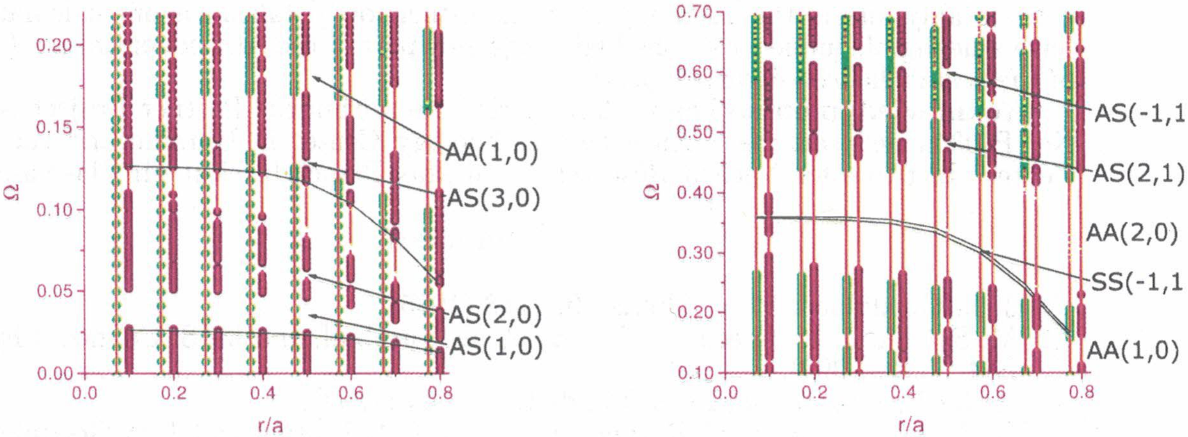

FIG. 1. The low-frequency part of the Alfvén and sound continua in W7-AS. Notations: $\Omega = \omega/(\bar{v}_A R_0)$, thick vertical lines, the Alfvén part of the continuum; thin red vertical lines, the sound part of the continuum; breaks in the lines, gaps. Green color, the case when the Alfvén-sound coupling is neglected; purple color, the realistic situation. The gaps are labeled by the coupling numbers (μ, ν) with addition of the letters AA for Alfvén gaps, SS for sound gaps and AS for Alfvén-sound gaps. Curves show the approximate position and size of the gaps estimated by the gap labeling procedure of the code.

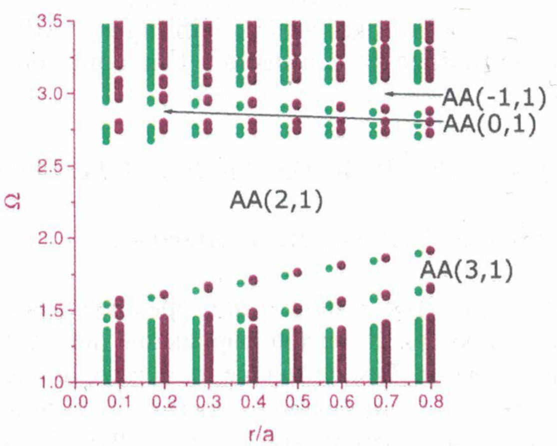

FIG. 2. The high-frequency part of the Alfvén spectrum in W7-AS. Notations are the same as in Fig. 1.

4. DISCUSSION

The reduced linear MHD equations derived in Sec. 2 are primarily intended for investigation of the effect of compressibility on Alfvén eigenmodes in the central part of a Helias stellarator. For this reason, omitting the pressure-gradient and current terms while retaining the compressibility terms is justified and makes the equations rather simple. Note that reduced equations valid for large m, n were earlier obtained with the use of the ballooning formalism in [13]. Those equations retained effects of the pressure gradient and the plasma current. However, an advantage of the equations obtained here is that they do not employ the ballooning formalism and, thus, are applicable to low-n perturbations. The continuous spectrum of the obtained equations exactly coincides with that of full MHD. Note that the slow-sound approximation used in Refs. [2,3] to calculate the effect of compressibility on Alfvén spectra in tokamaks does not seem applicable to Helias stellarators, where coupling with $\nu \neq 0$ is strong.

The code COBRAS capable of calculating the MHD continuum in stellarators with taking into account the compressibility has been developed. First results of the code for a realistic W7-AS configuration reveal the existence of absolute Alfvén, sound and Alfvén-sound gaps – features expected from analysis of two-mode coupling. The compressibility considerably affects the TAE gap (even at rather low β taken for our calculations) but have a negligible influence on the high-frequency part of the Alfvén continuum (the region of the widest helicity-induced gaps).

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РЕДУКОВАНІ МГД РІВНЯННЯ ДЛЯ АЛЬФВЕНІВСЬКИХ МОД У СТЕЛАРАТОРІ

О. П. Фесенюк, Я. І. Колесниченко, Г. Вобіґ, Ю. В. Яковенко

Виведено редуковані магнітогідродинамічні (МГД) рівняння, в яких враховується стисливість плазми, але виключаються швидкі магнітозвукові хвилі. З метою спрощення не розглядаються доданки, пов'язані з градієнтом тиску та струмом плазми. Тим не менше, показано, що неперервний спектр отриманих рівнянь повністю збігається з неперервним спектром повної системи магнітогідродинамічних рівнянь. Представлено перші результати коду COBRAS (COntinuum BRanches of Alfven and Sound waves – неперервні гілки альфвенівських та звукових хвиль), призначеного для обрахунку зачеплених альфенівського та звукового континуумів. Досліджено вплив стисливості на альфвенівський континуум стелараторів серії Вендельштайн.