METRIC COEFFICIENTS FOR A STELLARATOR CONFIGURATION

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The metric coefficients are analytically calculated for a non-axisymmetric stellarator configuration with a complex shape of the magnetic axis and the magnetic field varying along the axis. Calculations are performed in a working coordinate system, which is a flux coordinate system with "straightened magnetic field lines", assuming that the magnetic axis is a closed spatial curve with given curvature and torsion. The cross sections of the magnetic surfaces are approximated by ellipses with both elongation and orientation of the main axes varying along the magnetic axis. Effects of triangularity and displacement of magnetic surfaces with respect to the magnetic axis are included in the analysis. Transition from the working coordinates to Boozer ones is carried out, and the metric coefficients are obtained in Boozer coordinates.

Introduction

Three-dimensional nature of the magnetic configuration of stellarators, especially when the magnetic axis is non-planar, results in a complicated form of the flux surfaces. Therefore, generally speaking, numerical calculations are required to determine the metric features of the system. Nevertheless, making some simplifying assumptions one can treat the problem analytically. In this work, we employ such an approach, assuming that the magnetic surfaces are a set of tori nested around a single magnetic axis.

The usual way to describe stellarator configurations is using flux coordinates (a,θ,φ), with a being a label of a magnetic surface and θ,φ some angles on the surface a=*const*. An appropriate choice of these angles depends on the problem under consideration.

So-called Boozer coordinates (a,θ_B,φ_B) [1] are most universal coordinates convenient for both analytical and numerical calculations. In such coordinates the flux (contravariant) representation of the equilibrium magnetic field is

$$2\pi\mathbf{B} = \nabla\Psi\times\nabla\theta_B - \iota\nabla\Psi\times\nabla\varphi_B, \tag{1a}$$

with $\Psi(a)$ being the toroidal magnetic flux and ι the rotational transform. The current (covariant) representation has the form:

$$2\pi\mathbf{B} = J(a)\nabla_a\theta_B + F(a)\nabla_a\varphi_B, \tag{1b}$$

where $J(a)$ and $F(a)$ are the toroidal and poloidal currents, and $\nabla_a = \nabla - \nabla a(\nabla a\nabla)/|\nabla a|^2$ is the surface gradient [2]. The Jacobian of the transformation from Cartesian to Boozer coordinates is

$$\sqrt{g^B} = \frac{\Psi'(F+\iota J)}{4\pi^2 B^2}. \tag{2}$$

Here, following [2], we use notations slightly different from original Boozer ones.

Interest in analytical treatment of metric coefficients is accounted for by the fact that they explicitly enter in many expressions if the curvilinear coordinates are used. For example, an equation of shear Alfvén eigenmodes in optimized stellarators of the Wendelstein line was derived in Ref. [3] in Boozer coordinates, using the ideal MHD approximation (the longitudinal component of the perturbed electric field $\widetilde{E}_\parallel = 0$) and taking $\widetilde{B}_\parallel = 0$. It has the form:

$$\hat{L}\nabla\cdot\left(\nabla_\perp\hat{L}\mathbf{\Phi}\right) + \omega^2 R_0^2\nabla\cdot\left(\frac{\nabla_\perp\mathbf{\Phi}}{\overline{v}_A^2 b^4}\right) = 0, \tag{3}$$

where $\hat{L} = \iota \frac{\partial}{\partial \theta_B} + \frac{\partial}{\partial \varphi_B}$, $\widetilde{\mathbf{E}} = -\nabla_\perp \Phi$, $\overline{v}_A = \overline{B}/\sqrt{4\pi\rho}$, $\overline{B} = B/b$, $b = 1 + \sum_{\mu,\nu} \in^{(\mu\nu)} \cos(\mu\theta_B - \nu N \varphi_B)$, μ and ν are integers, N is the number of the magnetic field periods, $\left|\in^{(\mu\nu)}\right| << 1$ are relative magnitudes of Fourier harmonics, and R_0 is the major radius of the torus. This equation is three-dimensional, and the dependence on θ_B, φ_B arises not only because of Fourier harmonics of B but also from the metric tensor coefficients.

In Ref. [3] the structure of the shear Alfvén continuum was studied on the basis of Eq.(3). However, the metric tensor coefficients were obtained in the mentioned work to the lowest order in the distance from the magnetic axis. In order to obtain the full picture of the Alfvén continuum, further work is required to find more exact expressions of tensor components, which is done in this paper.

Model

For purposes of calculation we take as our initial coordinate system a system associated with the magnetic axis of an equilibrium configuration, assuming that the magnetic axis is a closed three-dimensional curve with given curvature $k(\varphi)$ and torsion $\kappa(\varphi)$. We denote $s = R\varphi$ the arc length along the axis, and we use $R = L/2\pi$, where L is the length of the magnetic axis. In the plane $s = const$ we introduce orthogonal coordinates (x, y), $x = \rho\cos\omega$, $y = \rho\sin\omega$, $\omega = \gamma + \delta$, where ρ is the distance from the magnetic axis to the magnetic surface, γ is the angle read off from the main normal to the magnetic axis. The coordinates (x, y) rotate around the magnetic axis. The square of an element of length in the coordinate system may be written in the form [4]:

$$dl^2 = dx^2 + dy^2 + 2\vartheta'(ydx - xdy)d\varphi + \left(R^2 h_s^2 + \left(x^2 + y^2\right)\vartheta'^2\right)d\varphi^2, \tag{4}$$

where $\vartheta' = \delta' - \kappa R$, $h_s = 1 - (kx\cos\delta + ky\sin\delta)$, primes mean derivatives with respect to φ.

As a zero approximation we take the "rounded" coordinate system (ρ, ω, φ) [5], with the cross-sections of the magnetic surfaces being approximated by ellipses. The ellipticity parameter $\varepsilon(\varphi) = \tanh\eta(\varphi)$ ($\varepsilon = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}$, $a_{1,2}$ being the main semi-axes of the ellipse) and the angle between the ellipse minor axis and the magnetic axis principal normal $\delta(\varphi)$ arbitrarily change along the system axis. Moreover, both the variation of the magnetic field along the axis, considered as given, and displacements of magnetic surfaces with respect to the magnetic axis are taken into account:

$$x = \frac{1}{\sqrt{b_0}} e^{-\eta/2}(\rho\cos\omega + \rho^2\xi_1), \quad y = \frac{1}{\sqrt{b_0}} e^{\eta/2}(\rho\sin\omega + \rho^2\xi_2). \tag{5}$$

Here the function $b_0(\varphi) = B_0(\varphi)/B_{0\max}$ characterizes the field distribution on the axis; ξ_1 and ξ_2 are the displacements of the magnetic surface.

As the next step we take a flux coordinate system (a, θ, φ) with "straightened magnetic field lines" and include the effect of triangularity of magnetic surfaces with respect to the magnetic axis in the analysis:

$$\begin{aligned} a^2 &= \rho^2 + 2(\alpha\cos 3\omega + \beta\sin 3\omega)\rho^3, \\ \omega &= \theta + a(\lambda_1\sin\theta + \mu_1\cos\theta + \lambda_3\sin 3\theta + \mu_3\cos 3\theta). \end{aligned} \tag{6}$$

Below we will refer to this coordinate system as the working one.

The parameters α, β introduced here determine completely, for a given ε, the shape of the cross section of a particular magnetic surface. The parameters $\lambda_1, \mu_1, \lambda_3, \mu_3$ are the rectification parameters to be determined. Substituting Eqs.(5)-(6) into Eq.(4), we find:

$$\lambda_1 = -2\xi_1 - \frac{k}{\sqrt{b_0}} \cos\delta e^{-\eta/2}, \quad \mu_1 = 2\xi_2 + \frac{k}{\sqrt{b_0}} \sin\delta e^{\eta/2}, \quad \lambda_3 = \alpha, \quad \mu_3 = -\beta. \tag{7}$$

Note that the parameters λ_3, μ_3 coincide with those calculated in [5], whereas λ_1, μ_1 explicitly depend also on the magnetic field distribution on the axis.

Metric coefficients in the working coordinates (a,θ,φ)

Substituting Eqs.(5)-(7) into Eq.(4), we can calculate the metric coefficients of the working coordinate system, keeping only the leading term and the first-order correction to it:

$$g_{11} = g_{11}^{(0)} + a g_{11}^{(1)}, \tag{8}$$

$$g_{11}^{(0)} = \frac{1}{b_0}(\cosh\eta - \sinh\eta\cos 2\theta),$$

$$g_{11}^{(1)} = \frac{2k}{b_0^{3/2}}\sinh\eta\left[(-\cos\theta + \cos 3\theta)\cos\delta e^{-\eta/2} + (\sin\theta + \sin 3\theta)\sin\delta e^{\eta/2}\right] +$$
$$+\frac{4}{b_0}\left[\cos\theta\left((\alpha - \xi_1)\sinh\eta + \xi_1 e^{-\eta}\right) + \sin\theta\left((\beta + \xi_2)\sinh\eta + \xi_2 e^{\eta}\right) + \right.$$
$$\left. + \cos 3\theta(-\alpha\cosh\eta + \xi_1\sinh\eta) + \sin 3\theta(-\beta\cosh\eta + \xi_2\sinh\eta)\right],$$

$$g_{22} = a^2 g_{22}^{(0)} + a^3 g_{22}^{(1)}, \tag{9}$$

$$g_{22}^{(0)} = \frac{1}{b_0}(\cosh\eta + \sinh\eta\cos 2\theta),$$

$$g_{22}^{(1)} = -\frac{2k}{b_0^{3/2}}\left[(\cos\theta\cosh\eta + \cos 3\theta\sinh\eta)\cos\delta e^{-\eta/2} + (\sin\theta\cosh\eta + \sin 3\theta\sinh\eta)\sin\delta e^{\eta/2}\right] +$$
$$+\frac{4}{b_0}\left[\cos\theta(\alpha\sinh\eta - \xi_1\cosh\eta) + \sin\theta(\beta\sinh\eta - \xi_2\cosh\eta) + \right.$$
$$\left. + \cos 3\theta(\alpha\cosh\eta - \xi_1\sinh\eta) + \sin 3\theta(\beta\cosh\eta - \xi_2\sinh\eta)\right]$$

$$g_{12} = g_{21} = a g_{12}^{(0)} + a^2 g_{12}^{(1)}, \tag{10}$$

$$g_{12}^{(0)} = \frac{1}{b_0}\sinh\eta\sin 2\theta,$$

$$g_{12}^{(1)} = -\frac{k}{b_0^{3/2}}\left[(\sin\theta e^{-\eta} + 2\sin 3\theta\sinh\eta)\cos\delta e^{-\eta/2} - (\cos\theta e^{\eta} + 2\cos 3\theta\sinh\eta)\sin\delta e^{\eta/2}\right] +$$
$$+\frac{4}{b_0}\left[\cos\theta(\xi_2 e^{\eta}) - \sin\theta(\xi_1 e^{-\eta}) + \cos 3\theta(-\beta\cosh\eta + \xi_2\sinh\eta) + \sin 3\theta(\alpha\cosh\eta - \xi_1\sinh\eta)\right],$$

For further calculations only the leading terms of the metric coefficients $g_{13} = g_{31}$, $g_{23} = g_{32}$ are important:

$$g_{13} = ag_{13}^{(0)} + a^2 g_{13}^{(1)}, \tag{11}$$

$$g_{13}^{(0)} = \frac{\eta'}{2b_0}(\sinh\eta - \cosh\eta\cos 2\theta) - \frac{b_0'}{2b_0^2}(\cosh\eta - \sinh\eta\cos 2\theta),$$

$$g_{23} = a^2 g_{23}^{(0)} + a^3 g_{23}^{(1)}, \tag{12}$$

$$g_{23}^{(0)} = \frac{\eta'}{2b_0}(\cosh\eta\sin 2\theta) - \frac{b_0'}{2b_0^2}(\sinh\sin 2\theta) - \frac{\vartheta'}{b_0}.$$

It should be mentioned that the leading terms of the metric coefficients presented above are mainly determined by the shape of the magnetic surfaces (varying elongation of the surface cross sections) and the variation of the magnetic field along the axis; only $g_{23}^{(0)}$ explicitly contains dependence on the torsion of the magnetic axis and rotation of ellipse. However, first-order corrections to the metric coefficients involve the curvature of the magnetic axis, the rotation of the ellipse, and the triangularity and displacements of magnetic surfaces.

$$g_{33} = R^2\left[1 - \frac{2ka}{\sqrt{b_0}}\left(\cos\theta\cos\delta e^{-\eta/2} + \sin\theta\sin\delta e^{\eta/2}\right)\right], \tag{13}$$

$$\sqrt{g} = \frac{aR}{b_0}\left[1 - \frac{2ka}{\sqrt{b_0}}\left(\cos\theta\cos\delta e^{-\eta/2} + \sin\theta\sin\delta e^{\eta/2}\right)\right]. \tag{14}$$

It should be mentioned that the obtained metric coefficients in the special case of a uniform configuration ($b_0 = 1$) are reduced to the coefficients calculated in [5].

Transition to Boozer coordinates (a, θ_B, φ_B)

In the working coordinate system (a, θ, φ) with "straightened magnetic field lines", the flux representation is

$$2\pi\mathbf{B} = \nabla\Psi \times \nabla\theta - \iota\nabla\Psi \times \nabla\varphi, \tag{15a}$$

whereas the current representation of the equilibrium magnetic field takes the form:

$$2\pi\mathbf{B} = J(a)\nabla_a\theta + F(a)\nabla_a\varphi + \nabla_a\phi, \tag{15b}$$

where $\phi(a,\theta,\varphi)$ is some unknown function which is periodic in θ and φ, and all other notations are the same as in Eqs.(1).

As Boozer coordinates are a special case of flux coordinates with "straightened magnetic field lines", transition from the working coordinates to Boozer ones can be found, following [2,6]:

$$\theta_B = \theta + \frac{\iota}{F + \iota J}\phi, \quad \varphi_B = \varphi + \frac{1}{F + \iota J}\phi, \tag{16}$$

where ϕ should be found with the use of the condition $\nabla \cdot \mathbf{B}_0 = 0$ and Eq.(15b) by solving the differential equation:

$$div\nabla_a\phi = -Jdiv\nabla_a\theta - Fdiv\nabla_a\varphi. \tag{17}$$

We seek a solution of Eq.(17) in the form

$$\phi(a,\theta,\varphi) = \phi_0(\varphi) + \phi_1(a,\theta,\varphi). \tag{18}$$

Substituting into Eq.(17) the expressions for metric coefficients g_{ik} and applying periodicity conditions to ϕ, we find:

$$\phi_0 = F\int_0^{\varphi}\left(\frac{b_0}{\langle b_0\rangle} - 1\right)d\varphi, \qquad \langle b_0\rangle = \frac{1}{2\pi}\int_0^{2\pi} b_0 d\varphi. \tag{19a}$$

In a uniform configuration ($b_0 \equiv 1$), ϕ_0 vanishes, and $\phi_1(a,\theta,\varphi)$, which is given by the equation

$$\frac{\partial\phi_1}{\partial\theta} = \frac{a^2}{R^2}\left[\varepsilon\cos 2\theta\left(J\frac{R^2}{a^2} + F\frac{\vartheta'}{\langle b_0\rangle}\right) + F\sin 2\theta\frac{\eta'}{2\langle b_0\rangle\cosh\eta}\right], \tag{19b}$$

becomes the dominant term. In any case, $\phi_1(a,\theta,\varphi)$ gives the correction of the second order in expansion over the distance from the magnetic axis and can be neglected in the framework of our approximation. Knowing the expressions for the function ϕ, we can now find expressions for Boozer coordinates (a,θ_B,φ_B). Following [2,6], transition from the working coordinates to Boozer ones is carried out as follows:

$$\theta_B = \theta + \iota\int_0^{\varphi}\left(\frac{b_0}{\langle b_0\rangle} - 1\right)d\varphi, \quad \varphi_B = \varphi + \int_0^{\varphi}\left(\frac{b_0}{\langle b_0\rangle} - 1\right)d\varphi. \tag{20}$$

Usually the variation of the magnetic field along the axis does not exceed few percents; therefore, the working coordinates with good accuracy coincide with the Boozer ones. Such a transition from the working coordinates to the Boozer ones does not change seriously the expressions for metric coefficients:

$$g_{11}^B = g_{11}, \quad g_{22}^B = g_{22}, \quad g_{12}^B = g_{12}; \tag{21a}$$

however,

$$g_{13}^B = \frac{1}{D}g_{13}, \quad g_{23}^B = \frac{1}{D}\left(g_{23} + \iota g_{22}(1-D)\right), \quad g_{33}^B = \frac{1}{D^2}g_{33}, \quad \sqrt{g^B} = \frac{1}{D}\sqrt{g}, \tag{21b}$$

with D being the Jacobian of the transformation from the working coordinates to the Boozer ones:

$$D = \frac{\partial(\theta_B,\varphi_B)}{\partial(\theta,\varphi)} = \frac{b_0}{\langle b_0\rangle}. \tag{22}$$

The metric coefficients derived were found to be in agreement with numerically calculated coefficients for the partially optimized stellarator Wendelstein 7-AS (W7-AS).

Note that the leading terms of metric coefficients given by expressions (21) coincide with the metric tensor coefficients calculated in [3] to the lowest order in the distance from the magnetic axis.

Conclusions

In this work, a non-axisymmetric stellarator configuration with a complex shape of the magnetic axis is investigated analytically, using the Boozer representation of the equilibrium magnetic field. Calculations are performed in a working coordinate system, assuming that the magnetic axis is a closed spatial curve with given curvature and torsion. The variation of the magnetic field along the axis is taken into account and also considered as given. The metric coefficients are calculated through expansion in powers of the distance from the magnetic axis.

Transition from the working coordinates to Boozer ones is carried out. It is shown that coordinate systems with good accuracy coincide one with another. However, the transition from the working coordinates to the Boozer ones slightly changes the expressions for some of metric coefficients. Preliminary analysis shows that the derived metric coefficients agree with numerically calculated ones for the partially optimized stellarator W7-AS.

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МЕТРИЧНІ КОЕФІЦІЄНТИ ДЛЯ ДОВІЛЬНОЇ СТЕЛАРАТОРНОЇ КОНФІГУРАЦІЇ

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Аналітично розраховано метричні коефіцієнти для неосесиметричної стелараторної конфігурації зі складною формою магнітної осі та варіацією магнітного поля вздовж неї. Розрахунки виконано в робочій системі координат з "випрямленими силовими лініями магнітного поля", вважаючи, що магнітна вісь являє собою замкнену просторову криву, яка характеризується заданими кривиною і крутінням. Поперечні перерізи магнітних поверхонь апроксимовано еліпсами, причому витягнутість і орієнтація головних осей еліпсів змінюються вздовж магнітної осі. Крім того, в модель включено трикутність та зміщення магнітних поверхонь відносно магнітної осі. Зроблено перехід від робочих координат до координат Бузера та пораховано метричні коефіцієнти в координатах Бузера.