ALFVÉN CONTINUUM IN STELLARATORS: GENERAL ANALYSIS AND SPECIFIC EXAMPLES

Ya. I. Kolesnichenko, H. Wobig [1], Yu. V. Yakovenko, J. Kißlinger [1]

[1] *Max-Planck Institut für Plasmaphysik, IPP-EURATOM Association, Garching bei München, Germany*

The Alfvén continuous spectrum in three-dimensional toroidal magnetic configurations is analyzed. Alfvén continua in Wendelstein-line stellarators (Wendelstein 7-AS and the designed Helias reactor HSR4/18) are calculated with the code COBRA, and the principal gaps in the Alfvén continua in these devices are found. It is shown that the shape of the plasma cross section strongly affects the Alfvén spectrum. Peculiarities of the Alfvén continuum of low-shear configurations are discussed. The frequencies of the calculated gaps in the Alfvén continuum in W7-AS are compared with the frequencies of the experimentally observed plasma oscillations.

1. INTRODUCTION

Historically, the continuous spectrum of the shear Alfvén waves in toroidal fusion devices first attracted considerable attention because of potential importance for plasma heating and current drive. Later it was discovered that discrete eigenmodes of the Alfvén spectrum, which reside in gaps of the Alfvén continuum [1] or near minima of continuum branches [2], can be destabilized by fast ions [3–5]. As the structure of the discrete eigenmodes is closely associated with that of the continuum, this discovery increased the interest in the properties of the Alfvén continuum, in particular, in stellarators [6–11]. The physical nature of the Alfvén continuum has been studied in many works (see, e.g., Refs. [12–17]). It has been found that the essential spectrum of ideal MHD (which includes the continuous spectrum, the dense discrete spectrum and eigenvalues of infinite multiplicity) arises from waves propagating one-dimensionally along rays (magnetic field lines) which never intersect the plasma boundary [16]. Typically, each mode forming the continuum is singular at a certain flux surface, where the condition of local wave resonance is fulfilled [13,14,17]. Thus, the continua of ideal MHD are objects characterized by certain radial structure, which is of importance for, e.g., continuum damping of Alfvén eigenmodes. The equation of local Alfvén resonance, which describes the essential spectrum of Alfvén waves, is a second-order differential equation involving only derivatives along a magnetic field line. The variation of the coefficients due to deviations from the cylindrical symmetry results in coupling of different harmonics, which produces gaps in the tokamak continuum near the points where the cylindrical continuum branches with different poloidal mode numbers but the same toroidal mode number would intersect in the absence of coupling [15]. Such gaps for different toroidal wave numbers were shown to possess envelopes, which can be labeled by the poloidal coupling number (i.e., the difference of the poloidal mode numbers of the intersecting continua) [18].

The structure of continua in three-dimensional configurations (for instance, in stellarators) is studied much less than in tokamaks. The coefficients of the equation of local resonance in asymmetric configurations are quasi-periodic along a field line (they are periodic along a field line only at those flux surfaces where the rotational transform, ι, is rational); therefore, it can be transformed to an equation, which is very similar to the Schrödinger equation with a quasi-periodic potential energy. Such equations have been studied in a number of works [7,19–21]. In particular, it is known that when the ratio of the periods is sufficiently irrational, the spectrum resembles the Cantor set with a system of gaps labeled by several integer numbers corresponding to the periods of the system (in the considered case, by the poloidal and toroidal coupling numbers). However, these results are relevant to the spectrum at a separate irrational value of ι and, therefore, say little about the spatial structure of the continuum, which, as mentioned above, is also of importance.

In Ref. [11] the Alfvén continuum in a Helias configuration was studied with the use of the elaborated code COBRA. The aim of this work is to present further results obtained with COBRA concerning the structure of the Alfvén continuum in advanced stellarator configurations (Wendelstein 7-AS and Helias reactor configurations).

The structure of the work is as follows. In Sec. 2, general properties of Alfvén continua in non-axisymmetric magnetic configurations are revealed. The concept of "absolute gaps" introduced in Ref. [11] is first explained on the basis of heuristic arguments and then confirmed by numerical calculations with the code COBRA. The calculations employ a relatively simple analytical approximation of the configuration of a Helias reactor HSR5/22 [24], which facilitates the analysis of results. Results of similar calculations for numerically obtained configurations of Wendelstein 7-AS and another reactor project, HSR4/18 [25], are presented in Sec. 3. Finally, in Sec. 4 the obtained results are summed up and discussed.

2. GENERAL PROPERTIES OF ALFVÉN CONTINUA IN STELLARATORS

We proceed from the following equation of the essential spectrum of shear Alfvén waves [1,11]:

$$\hat{L}\left(h_g\hat{L}\Phi\right) + \frac{\omega^2 R_0^2}{\bar{v}_A^2} h_c\Phi = 0, \qquad (1)$$

where Φ is the wave function (scalar potential of the perturbation), ω is the wave frequency, $h_g = g^{\psi\psi}/\bar{g}^{\psi\psi}$, $h_c = g^{\psi\psi}\bar{B}^4/(\bar{g}^{\psi\psi}B^4)$, $g^{\psi\psi} \equiv |\nabla\psi|^2$ is a component of the contravariant metric tensor of a Boozer coordinate system (ψ, θ, ϕ) [22], the overbar denotes the flux-surface average, $\hat{L} = \iota\partial/\partial\vartheta + \partial/\partial\varphi$, v_A is the Alfvén velocity, $R_0 = L/(2\pi)$, L is the length of the magnetic axis. Equation (1) is completed with the natural boundary condition of periodicity. It describes local resonance of Alfvén waves at a "separate" flux surface and includes only differentiation within the flux surface (by angular variables) with ψ considered as a parameter. Therefore, the eigenvalues of the equation are functions of ψ and produce branches of the continuum as ψ is varied. It has been shown that in non-axisymmetric configurations (at least, in the case when the equation reduced to the Schrödinger form) the equation possesses, in addition to the continuous spectrum, the discrete dense spectrum [21]. However, we have not noticed any evidence of the presence of the discrete dense spectrum in our calculations. We expand the coefficients of Eq. (1) in Fourier series,

$$h_{g,c} = 1 + \frac{1}{2}\sum_{\mu,\nu=-\infty}^{\infty} \epsilon_{g,c}^{(\mu\nu)} \exp(i\mu\theta - i\nu N\phi) \qquad (2)$$

and assume that an arbitrary wave harmonic with the poloidal and toroidal mode numbers (m, n) can interact only with harmonics with the mode numbers $(m+\mu, n+\nu N)$, where μ and ν are integers, N is the number of the field periods, which enables us to take Φ in the following form,

$$\Phi = \sum_{\mu,\nu=-\infty}^{\infty} \Phi_{\mu\nu} \exp[i(m+\mu)\theta - i(n+\nu N)\phi]. \qquad (3)$$

Then Eq. (1) is reduced to an eigenvalue problem of the form

$$\sum_{\mu,\nu=-\infty}^{\infty} \mathcal{G}_{\mu_*,\nu_*;\mu,\nu}\Phi_{\mu\nu} = \omega^2\frac{R_0^2}{\bar{v}_A^2}\sum_{\mu,\nu=-\infty}^{\infty} \mathcal{C}_{\mu_*,\nu_*;\mu,\nu}\Phi_{\mu\nu} \qquad (4)$$

with infinite Hermitian matrices $\mathcal{G}$ and $\mathcal{C}$ (see Ref. [11] for details). The matrix elements depend on m and n only through $\tilde{k} \equiv m\iota - n$. Thus, if modes with different mode numbers are characterized by the same values of $\tilde{k}$ for certain ι, they produce identical patterns of continuum branches at that ι. For this reason, it is convenient to consider the problem in terms of $\tilde{k}$ and do not check all the time whether any given value of $\tilde{k}$ corresponds to any integer m and n, which can be interpreted also as withdrawing the periodicity condition and considering waves in an infinite periodic structure.

In a cylindrical plasma the continuum is a dense web of branches $\omega_{mn} = k_{mn}\bar{v}_A = |m\iota - n|\bar{v}_A/R_0$, which sweeps all values of ω. Any factor breaking the cylindrical symmetry (e.g., toroidicity) results in a gap in the continuum, which we will refer to as an "absolute" gap. To explain intuitively how the absolute gap appears, let us assume that the mode coupling is weak and consider the interaction of only two modes with the mode numbers (m_1, n_1) and $(m_2 = m_1 + \mu, n_2 = n_1 + \nu N)$. On truncating the matrices in Eq. (4) to the corresponding 2×2 submatrices, the eigenvalue problem can be solved exactly. When the Fourier coefficients $\epsilon_g^{(\mu\nu)}$ and $\epsilon_c^{(\mu\nu)}$, which we refer to as coupling parameters, do not vanish, the cylindrical branches are broken near the point of their intersection (cf. Ref. [15,11]), producing a gap of the width of $|\epsilon_g^{(\mu\nu)} + \epsilon_c^{(\mu\nu)}|\omega^{(\mu\nu)}(\iota)$, where

$$\omega^{(\mu\nu)}(\iota) = |N\nu - \mu\iota|\frac{\bar{v}_A}{2R_0} \tag{5}$$

is the frequency of the intersection point expressed in terms of ι. As both $\omega^{(\mu\nu)}(\iota)$ and $\epsilon^{(\mu\nu)}$ do not depend on the mode numbers, the envelopes of the higher and lower branches corresponding to various (m_1, n_1) and $(m_2 = m_1 + \mu, n_2 = n_1 + \nu N)$ (with μ and ν fixed) enclose a frequency gap stretched in the radial direction, which we will refer to as an "absolute gap". No continuum branch can cross the absolute gap. Indeed, an arbitrary cylindrical branch $\omega = \omega_{mn}(\iota)$ intersecting the curve $\omega = \omega^{(\mu\nu)}(\iota)$, as one can show, necessarily meets another branch $\omega = \omega_{m\pm\mu, n\pm\nu N}$, where the sign depends on the sign of k_{mn}, and is broken by the coupling. Similar considerations involving branches of sound spectra imply that in addition to the absolute Alfvén gaps considered above, coupling produces also sound gaps, where branches of the Alfvén continuum are present, but branches of slow (sound) continuum are absent, and Alfvén-sound gaps, where all continua disappear [23].

Although the above arguments are heuristic as they were inferred from considering only two interacting harmonics, they are supported by numerical calculations with the computer code COBRA [11]. On solving the eigenvalue problem given by Eq. (4) with the matrices truncated to a finite size, the code selects several eigenvalues which suffer from the truncation to the least extent. Figure 1 shows results of calculations for the 5-period Helias reactor HSR5/22 [24]. In this calculation, an paraxial analytical approximation for h_g and h_c was used (see Ref. [11] for details). The frequencies in Fig. 1 are normalized by the local Alfvén frequency (thus, the effect of the inhomogeneity of the plasma density is not displayed). Dots represent the continuum obtained by scanning $\tilde{k}$. The gaps [voids in the continuum, which are labeled in the figure by the coupling numbers, (μ, ν)] appear near the places given by Eq. (5) but may be shifted, especially, in the vicinity of other wide gaps. The width of the gaps is usually narrower than that expected from the two-mode consideration. One can see that the gaps resulting from plasma shaping are more pronounced than those resulting from inhomogeneity of B. Another interesting feature is the appearance of "combination gaps", i.e., the gaps that result from joint action of several harmonics rather than from a single harmonic of h_g and/or h_c. The very simple approximation of h_g and h_c adopted here makes it easy to notice such gaps. An example is the $(4,2)$ gap: There is no $(4,2)$ Fourier harmonic in both h_g and h_c. Therefore, the gap probably appears as the combination $(4,2) = (2,1) + (2,1)$. Other examples are $(4,3) = (2,1) + (2,2)$, $(6,3) = (2,1) + (2,1) + (2,1)$, etc. As scanning is performed with a lesser step, more gaps are found; thus, the structure of the continuum resembles a Cantor set.

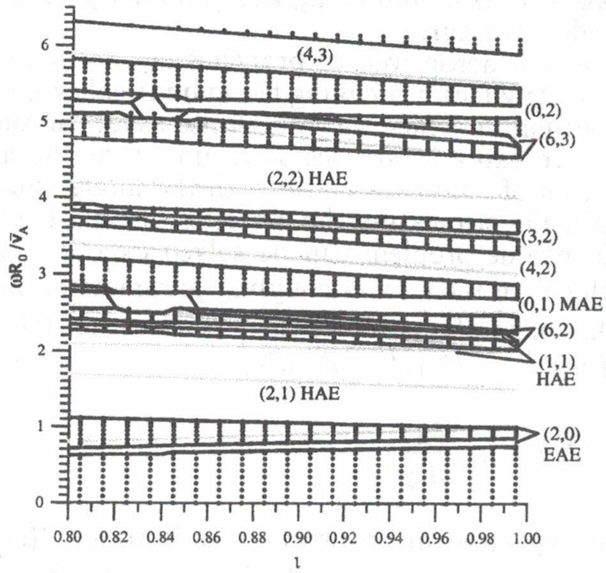

FIG. 1. The calculated continuous spectrum of Alfvén waves in HSR5/22 for an analytical model of the metric tensor and the magnetic field [11]). The gaps are labeled by the coupling numbers (μ,ν) responsible for the formation of each gap. Dots, the calculated frequencies of the continuum spectra obtained as $\tilde{k}$ of the dominant harmonic was scanned; thin lines, the calculated "banks" of the absolute gaps for various coupling numbers (μ,ν); wide grey lines, the lines $\omega = \omega^{(\mu,\nu)}(\iota)$, i.e., the places where the gaps would be located for the infinitely small coupling parameters ϵ_g and ϵ_c.

3. ALFVÉN CONTINUA IN HSR4/18 AND W7-AS

The code COBRA was used for calculating the Alfvén continua in two numerically calculated equilibria in Wendelstein-line stellarators. In these calculations, $g^{\psi\psi}$ and B were obtained numerically so that all substantial Fourier harmonics were taken into account. Figure 2 shows the Alfvén continuum in the 4-period Helias reactor HSR4/18 [25]. As the configurations of the 5-period and 4-period reactors are qualitatively similar, the main gaps in Fig. 2 are the same as in Fig. 1, but the number of gaps is larger because the analytical model used in Fig. 1 does not take into account all Fourier harmonics of the configuration. The increased number of gaps seems to be the reason why the algorithm of demarcating the gaps implemented in COBRA fails to work for this configuration. To label the gaps, the calculations were repeated with the coupling parameters $\epsilon_{g,c}^{(\mu\nu)}$ artificially decreased, after which the demarcation algorithm could work. Then main gaps in Fig. 2 were labelled by analogy.

The Alfvén continuum in the shot 43348 of Wendelstein 7-AS (W7-AS) is presented in Fig. 3. The Fourier spectrum of the magnetic field and the flux-surface shape in this device is more complicated, resulting in wider and more numerous gaps in the continuum. The gaps were labeled with the use of the same method (artificial decrease of coupling). Like in the above examples, the widest gaps, $(2,1)$ and $(3,1)$, result from non-circularity of the plasma (in this case, elongation and triangularity). The continuum in the frequency range between the $(-1,1)$ and $(4,1)$ gaps turns into thin threads between very wide gaps. All separate continuum branches are squeezed into these threads, and it is hardly possible to trace them anymore. This fact must have serious consequences, in particular, for GAE modes [2], which owe their existence to local minima of continuum branches.

A curious feature of several gaps $[(4,1), (5,1), (6,1)]$ is that the gap width vanishes at some radius. The most probable explanation of such behavior consists in the competition of direct coupling (dominating at large radii but tending to zero as $r^{\mu-2}$ at $r\to 0$) and combination coupling, which occasionally act in the opposite directions. For example, the characteristic width of the combination gap $(4,1) = (2,1) + (2,0)$ is $\sim \epsilon^{(21)}\epsilon^{(20)}$, which is less than $\epsilon^{(41)}$ at the periphery but exceeds $\epsilon^{(41)}$ in the center, where $\epsilon^{(41)} \propto r^2$, $\epsilon^{(21)} \propto r^0$, $\epsilon^{(20)} \propto r^0$.

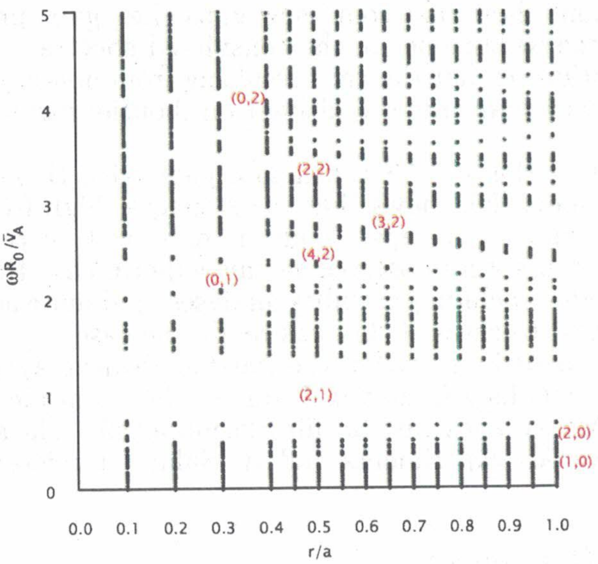

FIG. 2. The same as in Fig. 1 but for the numerically calculated configuration of HSR4/18.

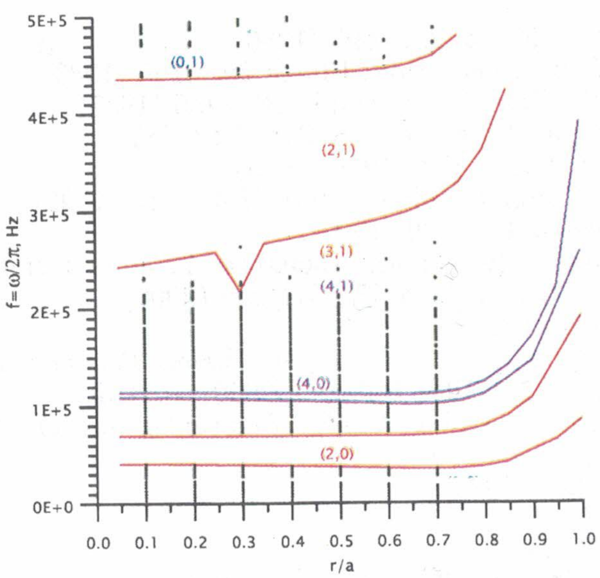

FIG. 3. The same as in Fig. 1 but for the numerically calculated plasma eqilibrium in the shot 43348 of W7-AS.

The shot 43348 characterized by strong neutral beam injection (NBI) into the plasma was selected for the calculations because in this shot strong bursts of oscillations with frequencies ranging from 20 to more than 250 kHz were observed in the Mirnov signal [26]. At a late stage, the oscillations looked like irregular bursts with the frequency spectrum consisting of several discrete lines. It is of interest to compare the frequencies of the lines with those of the continuum gaps. It turns out that the strongest line of the observed Mirnov spectrum (about 225–235 kHz) coincides with the (3, 1) and (4, 1) gaps. On the other hand, there is no considerable gap with the frequency of about 190 kHz, which is observed in the experiment, too. Probably, the nature of the spectral lines is different, but the ultimate answer cannot be given without stability analyses.

4. DISCUSSION

We have found that the Alfvén continuum in stellarators, like in tokamaks, is cut by gaps, which is a prerequisite for the existence of discrete eigenmodes. However, the absence of axial symmetry results in coupling of modes with different toroidal mode numbers and increases the variety of gaps in the stellarators. The gaps form a system of crossing bands and are likely to make the continuum a Cantor-type set at each irrational ι (cf. Refs. [21,19,20]). Our calculations show that some new gaps (i.e., gaps produced due to toroidal coupling) are among the widest ones in the considered spectra. Another observation following from our calculations is that the gaps resulting from non-circularity of the plasma shape are more pronounced than those resulting from Fourier harmonics of the magnetic field.

The comparison of the frequencies of the W7-AS continuum gaps with those of the oscillations observed in an NBI experiment [26] shows that the strongest high-frequency spectral line coincides with the $(3,1)$ and $(4,1)$ gaps. Thus, it may well be that new types of eigenmodes specific for stellarators were observed in this experiment. However, thorough investigation of such eigenmodes, including stability analyses, and more accurate studies of experimental data are required to check if this was really the case.

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References

[1] C. Z. Cheng and M. S. Chance, Phys. Fluids **29**, 3695 (1986).
[2] S. M. Mahajan, D. W. Ross, and G.-L. Chen, Phys. Fluids **26**, 2195 (1983).
[3] Y. M. Li, S. M. Mahajan, and D. W. Ross, Phys. Fluids **16**, 1466 (1987).
[4] J. Weiland, M. Lisak, and H. Wilhelmsson, Phys. Scr. **T16**, 53 (1987).
[5] G. Y. Fu and J. W. Van Dam, Phys. Fluids B **1**, 1949 (1989).
[6] N. Nakajima, C. Z. Cheng, and M. Okamoto, Phys. Fluids B **4**, 1115 (1992).
[7] A. Salat, Plasma Phys. Control. Fusion **34**, 1339 (1992).
[8] A. Salat and J. A. Tataronis, "Shear Alfvén mode resonances in nonaxisymmetric toroidal low-pressure plasmas", Rep. IPP III/263, IPP III/264 (2000).
[9] C. Nührenberg, Phys. Plasmas **6**, 137 (1999).
[10] C. Nührenberg, in *ISSP-19 "Piero Caldirola", Theory of Fusion Plasmas, J. W. Connor, O. Sauter, and E. Sindoni (Eds.)* (SIF, Bologna, 2000), p. 313.
[11] Ya. I. Kolesnichenko, V. V. Lutsenko, H. Wobig, Yu. V. Yakovenko, and O. P. Fesenyuk, Phys. Plasmas **8**, 491 (2001).
[12] J. P. Goedbloed, Phys. Fluids **18**, 1258 (1975).
[13] Y. Pao, Nucl. Fusion **15**, 631 (1975).
[14] E. Hameiri, Phys. Fluids **24**, 562 (1981).
[15] C. E. Kieras and J. A. Tataronis, J. Plasma Phys. **28**, 395 (1982).
[16] E. Hameiri, Comm. Pure Appl. Math. **XXXVIII**, 43 (1985).
[17] A. Salat and J. A. Tataronis, Phys. Plasmas **6**, 3207 (1999).
[18] M. S. Chu, J. M. Greene, L. L. Lao, A. D. Turnbull, and M. S. Chance, Phys. Fluids B **4**, 3713 (1992).
[19] E. I. Dinaburg and Ya. G. Sinai, Funct. Anal. Appl. **9**, 279 (1975).
[20] B. Simon, Adv. Appl. Math. **3**, 463 (1982).
[21] A. Salat, Phys. Rev. A **45**, 1116 (1992).
[22] A. H. Boozer, Phys. Fluids **24**, 1999 (1981).
[23] O. P. Fesenyuk, Ya. I. Kolesnichenko, H. Wobig, Yu. V. Yakovenko, "Reduced MHD equations for Alfvén eigenmodes in stellarators", this workshop.
[24] H. Wobig, C. D. Beidler, J. Kißlinger, E. Harmeyer, and F. Herrnegger, in *Fusion Energy 1998, 17th International Atomic Energy Agency Conference Proceedings, Yokohama, 1998*, Vol. 4 (International Atomic Energy Agency, Vienna, 1999), p. 1235.
[25] C. D. Beidler *et al.*, "The Helias reactor", in *Fusion Energy 2000, 18th International Atomic Energy Agency Conference Proceedings, Sorrento, 2000*, Report IAEA-CN-77/FT/4 (to be published).
[26] A. Weller *et al.*, in *Fusion Energy 1998, 17th International Atomic Energy Agency Conference Proceedings, Yokohama, 1998* (International Atomic Energy Agency, Vienna, 1999), Vol. 1, p. 295.

АЛЬФВЕНІВСЬКИЙ КОНТИНУУМ У СТЕЛАРАТОРАХ: ЗАГАЛЬНИЙ АНАЛІЗ ТА КОНКРЕТНІ ПРИКЛАДИ

Я. І. Колесниченко, Г. Вобіґ, Ю. В. Яковенко, Й. Кіcслінґер

Розглянуто альфвенівський неперервний спектр у тривимірних тороїдальних магнітних конфігураціях. Кодом COBRA пораховано альфвенівський континуум у стелараторах серії Вендельштайн (Вендельштайні 7-AS та проектованому Геліас-реакторі HSR4/18) та знайдено головні щілини альфвенівських континуумів у цих пристроях. Показано, що форма перерізу плазми сильно впливає на альфвенівський спектр. Обговорюються особливості альфвенівського континуума в конфігураціях із низьким широм. Частоти порахованих щілин альфвенівського континуума у W7-AS порівнюються з частотами експериментально спостережених коливань плазми.